J. Fluid Mech. (1982), *vol.* 114, *pp.* 123–156
Printed in Great Britain

Internal waves in a contained rotating stratified fluid

By SUSAN FRIEDLANDER

Department of Mathematics, University of Illinois, Chicago Circle, Chicago, Illinois 60680

AND WILLIAM L. SIEGMANN

Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181

(Received 23 April 1980 and in revised form 31 March 1981) 12181

Small-amplitude time-dependent motions of a uniformly rotating, density-stratified, Boussinesq non-dissipative fluid in a rigid container are examined for the case of the rotation axis parallel to gravity. We consider a variety of container shapes, along with arbitrary values for the (constant) Brunt–Väisälä and rotation frequencies. We demonstrate a number of properties of the eigenvalues and eigenfunctions of square-integrable oscillatory motions. Some of these properties hold generally, while others are shown for specific classes of containers (such as with symmetry about the container axis). A full solution is presented for the response of fluid in a cylindrical container to an arbitrary initial disturbance. Features of this solution which are different from the cases of no stratification or no rotation are emphasized. For the situation when Brunt–Väisälä and rotation frequencies are equal, characteristics of the oscillation frequencies and modal structures are found for containers of quite general shape. This situation illustrates, in particular, effects which are possible when rotation and stratification act together and which have been overlooked in previous investigations that assume that the vertical length scale is much smaller than the horizontal scales.

1. Introduction

Fluids which are rotating or stratified are known to possess a fascinating variety of wave motions. In this paper we are concerned with several aspects of small-amplitude oscillations in fluids which are both rotating and stratified. Discussion of such waves in unbounded media is found in a number of texts, for instance, Kamenkovich (1977), Krauss (1966) and Phillips (1977). However, the influence of boundary surfaces on these waves has been considered only recently, as is discussed below. Furthermore, we are unaware of any published investigations of the general type and characteristics of waves in rotating stratified fluids which are enclosed in a container.

This paper has several purposes. One is to indicate some principal properties of the frequencies and modal structures of the oscillations. We consider here only Boussinesq fluids with linear stratification, i.e. constant Brunt–Väisälä frequency N, and we report on other stratifications subsequently (see also §7). We defer to a later paper discussion of viscous effects. A prominent feature of our investigation is the inclusion of arbitrary values of N and of the Coriolis parameter f. In this way we attempt to assess the relative importance of stratification and rotation on the motions that we discuss. For instance, we indicate a number of properties of particular oscillations to which we refer as class II modes, with frequencies λ in the range $0 < \lambda^2 \leqslant \min(f^2, N^2)$.

Class II waves resulting from variations in the boundary surface(s) of a homogeneous fluid have been investigated in many papers (see LeBlond & Mysak (1979) and Mysak (1980*a*) for references and commentary). However, very few papers have considered such waves in a continuously stratified fluid, and none examine the problem in the complete N^2/f^2 parameter range. In particular, Wang & Mooers (1976), Clarke (1977) and Huthnance (1978) investigate the class II waves which propagate in a direction parallel to a horizontal shoreline in a uniformly rotating stratified fluid, when the fluid depth may or may not vary in the (off-shore) direction normal to the coast. Class II waves with vertical nodes that arise when the mean fluid depth is uniform are known as internal Kelvin waves because of the dynamical similarity to the (barotropic) Kelvin wave which exists for a homogeneous rotating fluid with a free surface. However, none of these papers study contained fluids. Further, all their analyses apply only to the limit of long waves, or small aspect-ratio limit, which is appropriate for some channel flows but not for most bounded fluids. Our results hold in the more complex situation where the horizontal and vertical scales are comparable, so that the oscillations do not preserve hydrostatic balance.

Another of our purposes is to characterize some effects of container geometry on rotating stratified fluid oscillations. Again, all previous investigations have considered only unbounded channels with some possible long shore variations (see Mysak 1980*a* for references), with primary focus on the barotropic modes. We discuss certain similarities and differences in the frequencies and structures of the oscillations which arise because of various container geometries. A further objective is to indicate how the energy in an arbitrary initial disturbance of a contained rotating stratified fluid is partitioned among the oscillations. Some of this energy excites the steady (frequency zero) mode, in a manner which has been discussed previously (Howard & Siegmann 1969). However, the distribution of the remaining energy among the time-dependent modes has not been considered elsewhere in the literature. We focus on the case of a cylindrical boundary surface when N^2/f^2 is arbitrary, but extensions are straightforward to other situations where the frequencies and structures of the oscillations can be obtained explicitly. The role of the class II modes in particular is described and illustrated.

Motivation for the investigation of these waves comes from both geophysical and laboratory situations. For example, oscillations in lakes could be studied by assuming a portion of the boundary is a free surface while the remainder is rigid. Similarly, this approach could be applied to obtain information about the oscillations in certain capes and bays. In this paper we restrict attention to the properties of waves in a completely rigid container, which is one experimentally relevant configuration, and omit the analogous development for the partially rigid case. As justification for our neglect of the free-surface effects, we note the conclusion of Kamenkovich (1977), that the approximation of a rigid lid filters out surface gravitational waves completely but hardly distorts internal gravitational waves.

We remark that some studies of internal waves in the ocean are concerned only with the behaviour of class I modes for which $\min(f^2, N^2) < \lambda^2 < \max(f^2, N^2)$. For example, a recent review (Garrett & Munk 1979) states that the 'permissible range of frequencies is $f \leqslant \omega \leqslant N$'. It is true that experimental data, for example that of Cairns & Williams (1976) for the power spectrum of vertical isotherm displacement, shows a sharp cut-off in energy at $|\lambda| = N$. However, significant energy levels are recorded for

$|\lambda| < f$. This suggests the importance of waves of subinertial frequencies, which consist of class II waves along with shelf waves, Rossby waves, etc. Indeed, in the deep ocean where N^2 and f^2 are comparable, waves of class II would be expected to be relatively more significant than those of class I. The importance of such waves is borne out by the large number of observations of low-frequency waves in coastal regions of the ocean. A review (Mysak 1980*b*) indicates the existence of such features in data from a great variety of the world's coasts. In addition, coastally trapped waves have been reported off some islands. The bulk of the oceanic observations have been for barotropic low-frequency waves, the existence of which depends on rotation but not stratification. However, some investigators have recognized the importance of the effects of stratification as well. For example, Smith (1978) suggests a significant role for baroclinic class II waves in the dynamics of the circulation off the western coast of South America. Some discussion of this particular paper, and its connection with a few of our results, will be made at the end of § 6.

It is to be expected that low-frequency baroclinic class II waves would be significant in many large lakes. This is because in winter N^2 is closer to f^2 than is the case in the upper ocean (see Eckart 1960). Also, the bounded geometry considered in this paper is a natural model of the closed boundary of lakes. For a lake, the horizontal length scales are sufficiently small for the influence of the boundaries on the interior oscillations to be very significant. In fact, Csanady (1976) has reported the presence of low-frequency waves in Lake Ontario. Wunsch (1973) has argued that baroclinic class II waves, of the particular type known as internal Kelvin waves, are a plausible mechanism for the production of a significant mean current. Such currents, which are counter-clockwise in the Northern hemisphere and may reverse with depth, have been widely observed in large lakes. Hamblin (1978) studied oscillations in a large lake using both numerical techniques and observations. He concluded that there was considerable evidence, in terms of frequencies and modal structure, for the occurrence of class II waves. Another example is provided by Ou & Bennett (1979), who studied the wind-induced motion in Lake Kinneret. They concluded that a large-amplitude internal Kelvin wave can reproduce well the primary response of the lake. Further, their theory can account for the principal features of the second-order mean current, if variable depth of the lake is incorporated into their model.

In § 2 we formulate the mathematical problem for the oscillations that we shall study. Particular attention is directed towards indicating the assumptions on length scales and non-dimensional parameters under which our analysis is valid. An eigenvalue problem is presented for the modal structures and frequencies. Assuming the existence of square-integrable oscillatory motions, we derive in § 3 a number of properties of the eigenvalue spectrum and eigenfunctions. For axisymmetric containers we present several more specific results. In § 4 detailed solutions for the class I, class II and transition oscillations (at the frequency separating class I and II behaviour) are presented for a right circular cylinder. Similar results are discussed for a sphere, and the two cases are compared to illustrate some effects of boundary geometry. In § 5, for the particular case $f^2 = N^2$, a description of both class I and II modes is provided for a container of arbitrary shape. In § 6 the solution for an arbitrary initial disturbance in the cylinder is found for all values of N and f. It is suggested that class II modes are typically excited with a non-negligible amplitude. Finally, a discussion of our results is provided in § 7. Amongst other matters, we explain the relationship between the mathematical problem we analyse and the corresponding one with rotation only and no stratification (Poincaré's problem), and the consequences of the differences are indicated.

2. Formulation

We consider the motions of a fluid rotating uniformly with angular velocity Ω about an axis parallel to gravity. We assume that the Boussinesq approximation is valid and, in this present paper, that the effects of viscous and thermal dissipation may be neglected. The equations of motion are given by

$$\mathbf{q}_t^* + \mathbf{q}^* . \nabla \mathbf{q}^* + f\hat{\mathbf{k}} \times \mathbf{q} = -\frac{1}{\bar{\rho}} \nabla P^* - \frac{g\rho^*}{\bar{\rho}} \hat{\mathbf{k}} + \frac{\rho^*}{\bar{\rho}} \frac{\Omega^2}{2} \nabla |\hat{\mathbf{k}} \times \mathbf{r}|^2, \tag{2.1}$$

$$\nabla . \mathbf{q}^* = 0, \quad \tau_t^* + \mathbf{q}^* . \nabla \tau^* = 0 \qquad (2.2), (2.3)$$

with the equation of state

$$\rho^* = \bar{\rho}[1 - \alpha(\tau^* - \bar{\tau})]. \tag{2.4}$$

The variables P^*, τ^*, ρ^* and $\mathbf{q}^*$ are the pressure, temperature, density and velocity relative to a co-ordinate system rotating with the fluid; g is the acceleration due to gravity in the direction of the unit vertical vector $\hat{\mathbf{k}}$; α is the coefficient of thermal expansion; $\bar{\rho}$ and $\bar{\tau}$ are constant mean values of the density and temperature; and $f = 2\Omega$ is the Coriolis parameter (in oceanographic problems $f = 2\Omega \sin$ (latitude)). The inviscid flow is required to satisfy the boundary condition

$$\mathbf{q}^* . \hat{\mathbf{n}} = 0 \quad \text{on } B \tag{2.5}$$

(B and $\hat{\mathbf{n}}$ are the container surface and the unit exterior normal vector), together with inhomogeneous initial conditions

$$\mathbf{q}^*(\mathbf{r}, 0) = \mathbf{q}^*(\mathbf{r}), \quad \tau^*(\mathbf{r}, 0) = \tau_0^*(\mathbf{r}). \tag{2.6}$$

We assume that the motions are small deviations from the motionless state P_s, τ_s and ρ_s. If we suppose that the Froude number $F_R = f^2L/g$ is sufficiently small so that the centrifugal force in (2.1) can be neglected, it follows that the functions P_s, τ_s and ρ_s depend only on the vertical co-ordinate z (L is a characteristic horizontal length scale). The linearized equations are obtained by setting

$$\mathbf{q}^* = \epsilon \mathbf{q}, \quad P^* = P_s + \epsilon \bar{\rho} P, \quad \rho^* = \rho_s + \epsilon \rho,$$

$$\tau^* = \tau_s - \epsilon \frac{d\rho_s}{dz} (\alpha \bar{\rho})^{-1} \tau,$$

where ϵ may be regarded as a Rossby number characterizing the magnitude of the initial perturbation. We simplify these equations by eliminating density in favour of temperature and by writing them in terms of variables $\mathbf{q}$, P and τ, to obtain

$$\mathbf{q}_t + f\hat{\mathbf{k}} \times \mathbf{q} = -\nabla P + N^2 \tau \hat{\mathbf{k}}, \tag{2.7}$$

$$\nabla . \mathbf{q} = 0, \quad \tau_t + \hat{\mathbf{k}} . \mathbf{q} = 0, \qquad (2.8), (2.9)$$

where the Brunt–Väisälä (or buoyancy) frequency is

$$N = \left\{-\frac{g}{\bar{\rho}}\frac{d\rho_s}{dz}\right\}^{\frac{1}{2}}.$$

In this paper we consider the case $N^2 > 0$ of stable linear density stratification. The perturbation variables satisfy the boundary condition of the form (2.5) and inhomogeneous initial conditions that are derived from (2.6), namely

$$\mathbf{q}(\mathbf{r}, 0) = \mathbf{q}_0(\mathbf{r}), \quad \tau(\mathbf{r}, 0) = \tau_0(\mathbf{r}). \tag{2.6a}$$

It is instructive to note the restrictions on the physical parameters that are implicit in our formulation of the problem given by equations (2.7)–(2.9) with (2.5) and (2.6*a*). In making the Boussinesq approximation, variations in fluid density are neglected as far as they influence inertia, but variations in buoyancy are retained. The specific conditions under which this approximation can be made (see Spiegel & Veronis 1960) are valid for most geophysical motions of interest. In neglecting diffusive effects, we have assumed that the Ekman number $E = \nu/fl^2$ and the parameter E/σ are both very small. Here l is the smallest of the vertical and horizontal length scales associated with the problem; ν is the coefficient of viscosity; and $\sigma = \nu/\kappa$ is the Prandtl number, in which κ is the coefficient of thermal diffusivity. Characteristic values of these parameters in the ocean or atmosphere are generally very small. On time scales of $O(f^{-1})$, the effects of diffusion in most geophysical contexts are negligible except possibly in thin boundary-layer regions. In assuming that the Rossby number ϵ is small, we are restricting our attention to small-amplitude oscillations where the magnitude of the resulting velocity fields are small compared with the rotational velocity fL. In this present paper, we consider only the case in which gravity is parallel to the axis of rotation. In assuming that the Froude number $F_R = f^2L/g$ is negligibly small, we have ignored the curvature of the potential surfaces due to centrifugal force. In the final section, we shall briefly discuss how the results described in this paper are modified when gravity and the rotation axis are not necessarily aligned. In particular, we will mention the geophysically relevant case of radially symmetric gravity in a sphere, as well as the effects of centrifugal force when the fluid is rapidly rotating.

Within the context of the approximations mentioned above, this mathematical model is valid for all values of N/f and for all values of the aspect ratio H/L (H is a characteristic vertical length scale). The only restriction is that the parameter f^2L/g remains small. In terms of physical applications, it is desirable to be able to permit a wide range of values of N/f. For example, in the thermocline region of the upper ocean, N/f is much greater than unity ($N/f = O(10^2)$). However, this is not true at great depths and, as Eckart (1960) remarks, one may scarcely conclude that the ratio is greater than unity everywhere. Observations in fresh-water lakes indicate the existence of strong stratification due to summer heating, giving values of N/f in September of 300. On the other hand, winter data from Lake Michigan suggests that N/f may be less than unity throughout a considerable part of its volume. Of course, the most realistic geophysical model would allow N to vary with depth. In a subsequent paper (see Friedlander & Siegmann 1981), many of the results of this present paper are extended to the case of variable N. As we discuss in §7, in the subsequent paper we also allow for curvature of the potential surfaces, which can be a significant factor when applying the results in a geophysical context.

We seek the solution to the above problem given by (2.5)–(2.9) in terms of a superposition of time-independent flow and time-dependent normal modes. The velocity $\mathbf{q}_g$ and the temperature τ_g of the steady portion are related to the pressure P_g by the geostrophic equations:

$$f\hat{\mathbf{k}} \times \mathbf{q}_g - N^2\tau_g\hat{\mathbf{k}} + \nabla P_g = 0, \quad \hat{\mathbf{k}}.\mathbf{q}_g = 0. \tag{2.10}$$

It has been shown (see, for example, Howard & Siegmann 1969) that P_g satisfies the potential vorticity equation

$$\frac{\partial^2 P_g}{\partial x^2} + \frac{\partial^2 P_g}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2 P_g}{\partial z^2} = f\hat{\mathbf{k}}.\nabla \times \mathbf{q}_0 + f^2\frac{\partial \tau_0}{\partial z} \tag{2.11}$$

together with certain boundary conditions. On the 'flat' portions of B, i.e. where $\hat{\mathbf{n}}$ is parallel to $\hat{\mathbf{k}}$, the boundary condition is that the initial temperature is conserved,

$$\frac{\partial P_g}{\partial z} = N^2\tau_0 \quad \text{on} \quad B_F. \tag{2.12}$$

On the rest of B where $|\hat{\mathbf{n}}.\hat{\mathbf{k}}| < 1$, let Γ be any closed curve bounded by the intersection of a plane z = constant with B. For each such curve Γ, one boundary condition is

$$\oint_\Gamma \left[\hat{\mathbf{n}}.\nabla_H P_g + \frac{f^2}{N^2}\hat{\mathbf{n}}.\hat{\mathbf{k}}\frac{\partial P_g}{\partial z}\right] ds = \oint_\Gamma [-f\hat{\mathbf{n}}.\hat{\mathbf{k}} \times \mathbf{q}_0 + f^2\hat{\mathbf{n}}.\hat{\mathbf{k}}\tau_0]\, ds, \tag{2.13}$$

where $\nabla_H P_g$ is the projection of ∇P_g in the horizontal plane z = constant and s is arc length. In addition, the boundary condition (2.5) requires that, for $|\hat{\mathbf{n}}.\hat{\mathbf{k}}| < 1$,

$$\hat{\mathbf{n}} \times \hat{\mathbf{k}}.\nabla P_g = 0. \tag{2.14}$$

It can be shown that (2.13) is equivalent to the conservation of the integral of temperature over the horizontal region bounded by Γ. An example of the solution of (2.11)–(2.14) is included in §6.

Once the steady component of the flow is determined, the time-dependent portion of the non-dissipative flow must be constructed as a solution of (2.5), (2.7)–(2.9) with

$$\mathbf{q}(\mathbf{r}, 0) = \mathbf{q}_0(\mathbf{r}) - \mathbf{q}_g(\mathbf{r}), \quad \tau(\mathbf{r}, 0) = \tau_0(\mathbf{r}) - \tau_g(\mathbf{r}). \tag{2.15}$$

We use the method of separation of variables to seek solutions of the form

$$(\mathbf{q}, \tau, P) = (\mathbf{Q}(\mathbf{r}), T(\mathbf{r}), \Phi(\mathbf{r}))\, e^{i\lambda t}. \tag{2.16}$$

The solution to (2.5), (2.7)–(2.9) and (2.15) would then be a superposition over λ of the modal solutions (2.16). Examples of the modal superposition will be given in §6. An alternative formulation of this initial-boundary-value problem is obtained if we substitute (2.16) into equations (2.5) and (2.7)–(2.9) to give

$$i\lambda\mathbf{Q} + f\hat{\mathbf{k}} \times \mathbf{Q} = -\nabla\Phi + N^2T\hat{\mathbf{k}}, \tag{2.17}$$

$$\nabla.\mathbf{Q} = 0, \quad i\lambda T + \hat{\mathbf{k}}.\mathbf{Q} = 0, \qquad (2.18), (2.19)$$

with

$$\mathbf{Q}.\hat{\mathbf{n}} = 0 \quad \text{on} \quad B. \tag{2.20}$$

We manipulate this system of vector equations to obtain the equation satisfied by the pressure Φ, namely

$$\nabla^2\Phi + \frac{N^2-f^2}{\lambda^2-N^2}\frac{\partial^2\Phi}{\partial z^2} = 0, \tag{2.21}$$

with the boundary condition

$$\hat{\mathbf{n}}.\nabla\Phi - \frac{f}{i\lambda}(\hat{\mathbf{n}}\times\hat{\mathbf{k}}).\nabla\Phi + \left[\frac{N^2-f^2}{\lambda^2-N^2}\right](\hat{\mathbf{n}}.\hat{\mathbf{k}})\frac{\partial\Phi}{\partial z} = 0 \quad \text{on} \quad B. \tag{2.22}$$

Equations (2.21) and (2.22) are the generalization of Poincaré's problem to incorporate the effects of stratification.

We must be circumspect in applying the results of our normal-mode analysis to arbitrary containers in view of several investigations of the homogeneous case $N^2 = 0$; cf. Stewartson & Rickard (1969), Stewartson (1971). Results for containers that are conical or concentric spheres suggest that the modal spectrum may be continuous and that the corresponding eigenfunctions may have singularities in the flow domain. Our general results are predicated on the existence of square-integrable solutions of (2.21) and (2.22). We confirm in later sections that this assumption is justified for particular geometries, as well as for more general geometry in the special case $N^2 = f^2$. In the final section, we comment briefly on the absence of continuous spectra.

3. General properties of oscillatory modes for $N^2 \neq f^2$

In this section we present several characteristics which must be satisfied by eigenfunctions $(\mathbf{Q}, T, \Phi)$ and eigenvalues λ of the system (2.17)–(2.20). We do not construct explicit solutions here, but rather assume that square-integrable eigenfunctions exist. That some container geometries exist for which this assumption is true is demonstrated in §4. We first prove general properties that hold for the solutions to the eigenvalue problem (2.17)–(2.20).

Property 1. For $N^2 \neq f^2$ a pressure gradient in the fluid is necessary to sustain an oscillatory mode.

Proof: Suppose $\nabla\Phi \equiv 0$ in (2.17). Using (2.19) in (2.17) yields

$$i\lambda(\mathbf{Q} - (N^2/\lambda^2)\,w\hat{\mathbf{k}}) + f\hat{\mathbf{k}}\times\mathbf{Q} = 0, \tag{3.1}$$

where we have written $w \equiv \hat{\mathbf{k}}.\mathbf{Q}$. Next, the scalar product of (3.1) and $\hat{\mathbf{k}}$ is

$$(\lambda^2 - N^2)\,w = 0, \tag{3.2}$$

and using (3.2) and the vector product of (3.1) and $\hat{\mathbf{k}}$ gives

$$(\lambda^2 - f^2)\,\mathbf{Q}_H = 0. \tag{3.3}$$

In (3.3) $\mathbf{Q}_H = \mathbf{Q} - w\hat{\mathbf{k}}$ is the projection of the velocity in the horizontal plane. From (3.2) and (3.3), it follows that $\mathbf{Q} = 0$ (and therefore also T, from (2.19) is zero), except possibly when $\lambda^2 = f^2$ or $\lambda^2 = N^2$. If $\lambda^2 = f^2$, then from (3.2), $w = 0$; let $\mathbf{Q}_H = u(x,y,z)\,\hat{\imath} + v(x,y,z)\,\hat{\jmath}$ in Cartesian co-ordinates. From (2.17), $v = \pm iu$, and by (2.18), it follows that u and v are harmonic functions with respect to the variables x and y. Hence (2.20) guarantees that $\mathbf{Q}_H \equiv 0$, a result obtained by Kudlick (1966) when $N^2 = 0$. If on the other hand $\lambda^2 = N^2$, then $\mathbf{Q}_H = 0$ from (3.3). From (2.18) and (2.20), w must vanish also.

The result of property 1, that the existence of an oscillatory mode assures a non-uniform pressure distribution in the container provided $N^2 \neq f^2$, justifies our formulation (2.21) and (2.22) of (2.17)–(2.20) in terms of pressure alone. We also make use of property 1 in some of our subsequent results.

Property 2. Eigenvalues λ are real.

Proof: Find the scalar product of (2.17) and $\mathbf{Q}^*$, where the asterisk denotes complex conjugate, then use (2.19) to eliminate T, and apply (2.18) to obtain

$$i\lambda|\mathbf{Q}|^2 + f\mathbf{Q}^*.\hat{\mathbf{k}}\times\mathbf{Q} + N^2(i\lambda)^{-1}|w|^2 = -\nabla.(\mathbf{Q}^*\Phi). \tag{3.4}$$

We integrate (3.4) over the volume V of the container and use Gauss' theorem to obtain

$$i\lambda\iiint_V|\mathbf{Q}|^2dv - i\lambda^*\frac{N^2}{|\lambda|^2}\iiint_V|w|^2dv = f\iiint_V\hat{\mathbf{k}}.\mathbf{Q}^*\times\mathbf{Q}\,dv. \tag{3.5}$$

The surface contribution that would appear in (3.5) vanishes because of (2.20). Next, find the scalar product of the conjugate of (2.17) with $\mathbf{Q}$ and repeat the above steps to obtain

$$i\lambda^*\iiint_V|\mathbf{Q}|^2dv + i\lambda\frac{N^2}{|\lambda|^2}\iiint_V|w|^2dv = f\iiint_V\hat{\mathbf{k}}.\mathbf{Q}\times\mathbf{Q}^*dv. \tag{3.6}$$

Adding (3.5) and (3.6) gives

$$i(\lambda-\lambda^*)\iiint_V\left\{|\mathbf{Q}|^2 + \frac{N^2}{|\lambda|^2}|w|^2\right\}dv = 0. \tag{3.7}$$

Using our assumption that modal eigenfunctions are square-integrable over the container volume, it follows from (3.7) that each λ is real.

Property 3. For $N^2 \neq f^2$, $\lambda^2 < \max(f^2, N^2)$.

Proof: Find the scalar product of (2.17) with its conjugate and hence obtain

$$|\nabla\Phi|^2 = \lambda^2|\mathbf{Q}|^2 + f^2|\mathbf{Q}_H|^2 + 2i\lambda f\mathbf{Q}.\hat{\mathbf{k}}\times\mathbf{Q}^* + N^4|T|^2 + i\lambda N^2(Tw^* - T^*w). \tag{3.8}$$

Use the conjugate of (3.4) in (3.8), together with (2.19) to eliminate T, to give

$$|\nabla\Phi|^2 = (f^2-\lambda^2)|\mathbf{Q}_H|^2 + \lambda^{-2}(N^4-\lambda^4)|w|^2 - i\lambda f\nabla.(\mathbf{Q}\Phi). \tag{3.9}$$

Then integrate (3.9) over V and use Gauss' theorem,

$$\iiint_V|\nabla\Phi|^2dv = \iiint_V\{(f^2-\lambda^2)|\mathbf{Q}_H|^2 + \lambda^{-2}(N^4-\lambda^4)|w|^2\}dv, \tag{3.10}$$

where again the surface contribution vanishes by (2.20). If either

(1) $N^2 < f^2$ and $\lambda^2 \geqslant f^2$

or

(2) $N^2 > f^2$ and $\lambda^2 \geqslant N^2$,

then the integrand on the right-hand side of (3.10) is non-positive. Using square-integrability of the modal eigenfunctions, in either of those cases it follows that

$|\nabla\Phi| = 0$. By property 1, we therefore conclude that no oscillatory modes can exist for $\lambda^2 \geqslant \max(f^2, N^2)$. The best frequency bound for a non-rotating fluid with vertically-varying stratification $N = N(z)$ is $\lambda^2 < N^2_{\max}$; property 3 can be generalized to the corresponding result in such a rotating variably-stratified fluid.

Although property 3 ensures that λ^2 is bounded from above by $\max(f^2, N^2)$, λ^2 is not bounded from below by $\min(f^2, N^2)$. Thus, we can expect to find solutions to the eigenvalue problem (2.21) and (2.22) for $0 < \lambda^2 < \max(f^2, N^2)$. The oscillatory modes for $N^2 \neq f^2$ divide naturally into two classes of solutions:

Class I, for which $\min(f^2, N^2) < \lambda^2 < \max(f^2, N^2)$ and (2.21) is hyperbolic; and

Class II, for which $0 < \lambda^2 < \min(f^2, N^2)$ and (2.21) is elliptic.

We note that treatment of the transition case $\lambda^2 = \min(f^2, N^2)$ proceeds differently, because manipulations in the derivation of (2.21) and (2.22) assume $\lambda^2 \neq f^2$ and $\lambda^2 \neq N^2$. We consider the transition case in properties 6 and 7.

Property 4. Let $(\mathbf{Q}_{(\lambda)}, T_{(\lambda)})$ and $(\mathbf{Q}_{(\mu)}, T_{(\mu)})$ be eigenfunctions corresponding to eigenvalues λ and μ, respectively. If $\lambda \neq \mu$, then

$$\iiint_V \{\mathbf{Q}_{(\lambda)} . \mathbf{Q}^*_{(\mu)} + N^2 T_{(\lambda)} T^*_{(\mu)}\}\, dv = 0. \tag{3.11}$$

Proof: Find the scalar product of (2.17) for λ-eigenfunctions with $\mathbf{Q}_{(\mu)}$, and that for the conjugate of (2.17) for μ-eigenfunctions with $\mathbf{Q}_{(\lambda)}$, and hence obtain

$$i\lambda \mathbf{Q}^*_{(\mu)} . \mathbf{Q}_{(\lambda)} + f\mathbf{Q}^*_{(\mu)} . \hat{\mathbf{k}} \times \mathbf{Q}_{(\lambda)} - N^2 T_{(\lambda)} w^*_{(\mu)} = -\mathbf{Q}^*_{(\mu)} . \nabla\Phi_{(\lambda)}, \tag{3.12}$$

$$-i\mu \mathbf{Q}^*_{(\mu)} . \mathbf{Q}_{(\lambda)} + f\mathbf{Q}_{(\lambda)} . \hat{\mathbf{k}} \times \mathbf{Q}^*_{(\mu)} - N^2 T^*_{(\mu)} w_{(\lambda)} = -\mathbf{Q}_{(\lambda)} . \nabla\Phi^*_{(\mu)}. \tag{3.13}$$

Adding (3.12) and (3.13) and using both (2.18) and (2.19) gives

$$i(\lambda-\mu)\, \mathbf{Q}^*_{(\mu)} . \mathbf{Q}_{(\lambda)} + i(\lambda-\mu)\, N^2 T^*_{(\mu)} T_{(\lambda)} = -\nabla . (\Phi_{(\lambda)} \mathbf{Q}^*_{(\mu)} + \Phi^*_{(\mu)} \mathbf{Q}_{(\lambda)}). \tag{3.14}$$

After integrating (3.14) and noting that the right-hand side vanishes by (2.20), we obtain (3.11) providing $\lambda \neq \mu$.

For $N^2 = 0$, (3.11) was derived by Greenspan (1968). As was the case for a homogeneous fluid, it is useful for solution of initial-value problems to find an alternative expression of (3.11) in terms of pressure eigenfunctions alone. With manipulations similar to those above, we can derive

$$(\lambda-\mu) \iiint_V \left\{\nabla\Phi_{(\lambda)} . \nabla\Phi^*_{(\mu)} + \frac{(f^2-N^2)(\lambda\mu+N^2)}{(\lambda^2-N^2)(\mu^2-N^2)}\, (\hat{\mathbf{k}} . \nabla\Phi_{(\lambda)})(\hat{\mathbf{k}} . \nabla\Phi^*_{(\mu)})\right\} dv = 0. \tag{3.15}$$

Appropriate modifications of (3.15) are necessary if either λ^2 or μ^2 equals N^2.

Property 5. Suppose the container B is symmetric about the z axis. Then any mode of class II necessarily propagates in the same direction as the mean rotation.

Proof: Define the operator

$$\tilde{\nabla} \equiv \nabla + \frac{N^2-f^2}{\lambda^2-N^2}\hat{\mathbf{k}}\frac{\partial}{\partial z}, \tag{3.16}$$

so that (2.21) and (2.22) may be written

$$\nabla . \tilde{\nabla}\Phi = 0, \quad \hat{\mathbf{n}} . \tilde{\nabla}\Phi + i\frac{f}{\lambda}(\hat{\mathbf{n}} \times \hat{\mathbf{k}}) . \nabla\Phi = 0 \quad \text{on} \quad B. \tag{3.17}$$

Multiplying the first of (3.17) by Φ^*, integrating over the container volume, and using Gauss' theorem yields

$$\iiint_V \tilde{\nabla}\Phi.\nabla\Phi^*\,dv = \iint_B \Phi^*\hat{\mathbf{n}}.\tilde{\nabla}\Phi\,d\Sigma. \tag{3.18}$$

Applying the boundary condition in (3.17) and simplifying, we obtain

$$\iiint_V \left\{|\nabla_H\Phi|^2+\frac{\lambda^2-f^2}{\lambda^2-N^2}|\Phi_z|^2\right\}dv = +\frac{f}{\lambda}i\iint_B \Phi^*(\hat{\mathbf{k}}\times\hat{\mathbf{n}}).\nabla\Phi\,d\Sigma. \tag{3.19}$$

Note that, for class II modes, the integrand on the left-hand side of (3.19) is non-negative. Moreover, from property 1, this integrand must be positive for $N^2 \neq f^2$. (A straightforward extension of property 1, using (3.10), shows that a pressure gradient must also exist for class II modes when $N^2 = f^2$ also.)

We let θ denote the (periodic) co-ordinate which increases in the direction of $\hat{\mathbf{k}}\times\hat{\mathbf{n}}$. Axisymmetry permits separation of the θ dependence, so that

$$\Phi = e^{ik\theta}\chi, \tag{3.20}$$

where χ is a function of two other co-ordinates appropriate to describe the region inside a cross-section of the container, and $k \geqslant 0$ without loss of generality. Using (3.20) in (3.19), it follows that

$$-\frac{f}{\lambda}\iint k|\chi|^2\,d\Sigma > 0, \tag{3.21}$$

which requires $k > 0$ and $f/\lambda < 0$. Therefore, class II modes propagate in the direction of increasing θ for $f > 0$.

We now prove two further properties that concern the existence of transition modes where $\lambda^2 = \min(f^2, N^2)$.

Property 6. Suppose $N^2 > f^2$ and let the boundary B consist of a single surface that is axisymmetric about the z axis. Then oscillations at the frequency $\lambda = -f$ have the form

$$\Phi(r,\theta,z) = g_k(z)\,r^k e^{ik\theta}, \tag{3.22}$$

where (r,θ,z) are cylindrical co-ordinates, k is a positive integer and g_k satisfies a boundary-value problem with eigenvalue $k(k+1)(N^2-f^2)/f^2$.

Proof: From (2.21) it follows that, for $\lambda^2 = f^2$,

$$\nabla_H^2\Phi = 0.$$

Thus

$$\Phi = \sum_{k=1}^{\infty} g_k(z)\,r^k e^{ik\theta} \tag{3.23}$$

(we note that $k = 0$ is excluded by property 5). From (2.17)–(2.19) the velocity and temperature for each wavenumber k can be written in terms of $g_k(z)$ as

$$w = \frac{if}{N^2-f^2}g_k'\,r^k e^{ik\theta} \tag{3.24}$$

and

$$T = \frac{g_k'\,r^k e^{ik\theta}}{N^2-f^2}, \tag{3.25}$$

and the velocity components (u, v) in the (r, θ) directions satisfy

$$u_r + \frac{1}{r}(1+k)\,u + \left[\frac{ik^2}{f} g_k r^{k-2} + \frac{if}{N^2-f^2} g_k'' r^k\right] e^{ik\theta} = 0 \tag{3.26}$$

and
$$v = -iu + \frac{k}{f} g_k r^{k-1} e^{ik\theta}. \tag{3.27}$$

The solution for u corresponding to a non-zero pressure field (cf. property 1) is

$$u = \frac{-i}{2f}\left[kr^{k-1} g_k + \frac{f^2 r^{k+1}}{(k+1)(N^2-f^2)} g_k''\right] e^{ik\theta}. \tag{3.28}$$

We assume that the axisymmetric surface B has the equation $r = F(z)$, where F is differentiable for $a < z < b$. The unit normal $\hat{\mathbf{n}}$ can then be written as

$$\hat{\mathbf{n}} = \frac{1}{(1+F'^2)^{\frac{1}{2}}}(1, 0, -F'). \tag{3.29}$$

From (3.27), (3.28) and (3.29), the boundary condition (2.20) becomes

$$F^2 g_k'' + 2(k+1)\,FF' g_k' + k(k+1)\,g_k(N^2-f^2)/f^2 = 0, \tag{3.30}$$

which can be written in a self-adjoint form as

$$(F^{2(k+1)} g_k')' + k(k+1)\,g_k(N^2-f^2)/f^2 = 0. \tag{3.31}$$

If $F(a) = 0$ (or $F(b) = 0$) the boundary condition for g_k is boundedness at $z = a$ (or $z = b$). Otherwise (3.24) and the boundary condition (2.20) require

$$g_k'(a) = 0 \quad \text{and} \quad g_k'(b) = 0.$$

Hence $g_k(z)$ is a solution to the above Sturm–Liouville problem, assuming $F(z) \neq 0$ in (a, b). In §4 we give the specific solutions for $g_k(z)$ in the particular geometries of the cylinder and the sphere.

It is a straightforward matter to generalize property 5 to the case where B consists of a pair of symmetric surfaces; for example, the boundary is given by co-axial circular cylinders. A generalization of this property may also be made for non-axisymmetric containers.

From property 5 we concluded that at least in the case of axisymmetric containers there are no class II modes with positive eigenvalues; hence there is no transition mode $\lambda = +f$.

Property 7. Suppose $N^2 < f^2$ and let the boundary B consist of a single axisymmetric surface. Oscillations at the frequency $\lambda = -N$ have the form

$$\Phi(r, \theta) = h_k(r)\, e^{ik\theta} \tag{3.32}$$

where k is a positive integer and $h_k(r)$ satisfies a boundary value problem with eigenvalue kf/N.

Proof: In the case $\lambda^2 = N^2$ the equations (2.17) and (2.19) can be manipulated to give

$$\frac{\partial \Phi}{\partial z} = 0, \quad u = \frac{-iN\Phi_r + f\Phi_\theta/r}{N^2-f^2}, \tag{3.33), (3.34}$$

$$v = \frac{-f\Phi_r - iN\Phi_\theta/r}{N^2-f^2}, \quad T = \frac{-iw}{N}. \tag{3.35), (3.36}$$

Substituting (3.33)–(3.36) into the divergence equation (2.18) and integrating with respect to z gives

$$w(r,\theta,z) = z\frac{iN}{N^2-f^2}\nabla_H^2\Phi + \frac{fw_0(r,\theta)}{N^2-f^2}. \tag{3.37}$$

Assume the surface B can be described by the equations $z = G_T(r)$ for $c < z < b$ and $z = G_B(r)$ for $a < z < c$ in which G_T and G_B are differentiable. Hence

$$\hat{\mathbf{n}}_T = \frac{1}{(1+G_T'^2)^{\frac{1}{2}}}(-G_T', 0, 1), \quad \hat{\mathbf{n}}_B = \frac{1}{(1+G_B'^2)^{\frac{1}{2}}}(-G_B', 0, 1). \tag{3.38}$$

Thus combining (3.34), (3.35), (3.37) and (3.38) in the boundary condition (2.20) gives the pair of equations

$$\left.\begin{aligned} G_T'(iN\Phi_r - f\Phi_\theta/r) + G_T iN\nabla_H^2\Phi + fw_0 &= 0,\\ G_B'(iN\Phi_r - f\Phi_\theta/r) + G_B iN\nabla_H^2\Phi + fw_0 &= 0.\end{aligned}\right\} \tag{3.39}$$

We substitute an expression for Φ of the form given by (3.32) into (3.39) and eliminate w_0 to give the following self-adjoint equation

$$(r(G_T-G_B)\,h_k'(r))' - [(G_T-G_B)\,k^2/r + (G_T-G_B)'kf/N]\,h_k(r) = 0. \tag{3.40}$$

Boundedness is required at $r = G_T^{-1}(c) = G_B^{-1}(c)$. The further conditions required are

(i) $w = 0$ at $r = G_T^{-1}(a)$ and $G_T^{-1}(b)$ provided $G_T^{-1}(a) \neq 0$, $G_T^{-1}(b) \neq 0$, or

(ii) boundedness at $r = 0$ if $G_T^{-1}(a) = 0$ or $G_T^{-1}(b) = 0$.

An important special case is $G_T = G_B$. Then (3.40) becomes

$$(rG_T h_k'(r))' - [G_T k^2/r + G_T' kf/N]\,h_k(r) = 0. \tag{3.41}$$

If $G_T > 0$ for $0 < r < G_T(b)$, this is a Sturm–Liouville problem and an infinite set of oscillations exist for each k when, for example, $G_T' < 0$. This property is illustrated in §4 in the example of the sphere.

If the top and bottom are both flat (for example, a cylinder or a truncated cone), (3.37) requires $w \equiv 0$. Thus from (3.37) the equation for $h_k(r)$ is

$$h_k'' + \frac{1}{r}h_k' - \frac{k^2}{r^2}h_k = 0. \tag{3.42}$$

Using (3.34) in the boundary condition (2.20) as applied to the sides of B gives

$$-Nh_k' + fkh_k = 0 \quad \text{on the sides of } B. \tag{3.43}$$

Since equations (3.42) and (3.43) are incompatible, we conclude that there are no oscillations at frequency $\lambda = -N$ in containers with two flat boundaries. This result is illustrated in the example of the cylinder in §4. Other generalizations are possible, for example, containers with one flat boundary, containers with multiple boundary surfaces.

We note that oscillations at frequency $\lambda = +N$ would have to satisfy (3.35) with the opposite sign on the kf/N term; such modes cannot occur in containers with $G_T' < 0$. This result is consistent with property 5, showing that there are no class II modes with positive frequencies in axisymmetric containers.

4. Oscillatory modes in the cylinder and sphere

4.1. *The cylinder*

In this section we first obtain explicit solutions for the eigenfunctions Φ and the eigenvalues λ describing internal waves in a rotating stratified fluid bounded by rigid cylindrical walls. Let (r, θ, z) denote cylindrical co-ordinates: the co-ordinates of the cylinder walls are chosen as $z = 0, h$ and $r = a$. In cylindrical co-ordinates equation (2.21) and boundary condition (2.22) become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2}+\frac{\lambda^2-f^2}{\lambda^2-N^2}\frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4.1}$$

with

$$\left[\frac{\lambda^2-f^2}{\lambda^2-N^2}\right]\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = 0, h \tag{4.2}$$

and

$$i\lambda\frac{\partial \Phi}{\partial r}+\frac{f}{r}\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{at} \quad r = a. \tag{4.3}$$

We shall examine the two types of solutions discussed in §3:

$$\text{class I}, \min(f^2, N^2) < \lambda^2 < \max(f^2, N^2);$$

$$\text{class II}, 0 < \lambda^2 < \min(f^2, N^2).$$

We shall also briefly consider the special cases $\lambda^2 = \min(f^2, N^2)$ and $\lambda^2 = f^2 = N^2$.

Class I solutions: $\min(f^2, N^2) < \lambda^2 < \max(f^2, N^2)$. The solution to the eigenvalue problem (4.1)–(4.3) is determined by separation of variables, giving modes of the form

$$\Phi = e^{ik\theta}\cos\frac{m\pi z}{h}J_k\left(\frac{\gamma_{mnk}r}{a}\right) \quad (m > 0,\ k \geqslant 0,\ \gamma_{mnk} \neq 0), \tag{4.4}$$

where J_k is the kth Bessel function of the first kind. The boundary condition (4.3) requires γ_{mnk} to be the nth positive solution to the transcendental equation

$$\lambda_{mnk}\gamma_{mnk}J'_k(\gamma_{mnk})+fkJ_k(\gamma_{mnk}) = 0, \tag{4.5}$$

where λ_{mnk} and γ_{mnk} are related by the equation

$$\lambda^2_{mnk} = \frac{\gamma^2_{mnk}N^2h^2+f^2(m\pi a)^2}{\gamma^2_{mnk}h^2+(m\pi a)^2}. \tag{4.6}$$

[Note that λ can be either positive or negative.]

These solutions are very similar to those in a homogeneous fluid except for the modified frequency range. Examination of (4.4)–(4.6) shows that class I waves exist if either $f = 0$ or $N = 0$. Some eigenvalues from (4.5) and (4.6) have been obtained numerically for $N^2 = 0$ by Kudlick (1966). When both rotation and stratification are present, the class I internal wave can be viewed as a rotational wave modified by stratification (or vice versa).

Class II solutions: $0 < \lambda^2 < \min(f^2, N^2)$. Equation (4.1) is elliptic and the solution to the eigenvalue problem is

$$\Phi = e^{ik\theta}\cos\frac{m\pi z}{h}I_k\left(\frac{\alpha_{mk}r}{a}\right) \quad (m > 0,\ k > 0,\ \alpha_{mk} \neq 0), \tag{4.7}$$

where I_k is the kth modified Bessel function of the first kind. The boundary condition (4.3) requires that α_{mk} satisfy the equation

$$\lambda_{mk}\alpha_{mk}I'_k(\alpha_{mk})+fkI_k(\alpha_{mk}) = 0, \tag{4.8}$$

with

$$\lambda^2_{mk} = \frac{\alpha^2_{mk}N^2h^2-f^2(m\pi a)^2}{\alpha^2_{mk}h^2-(m\pi a)^2}. \tag{4.9}$$

We next investigate the existence of solutions λ_{mk} and α_{mk} for the coupled system of equations (4.8) and (4.9). Using recurrence relations satisfied by I_k (see, for example, Relton 1946), equation (4.8) can be rewritten in the form

$$\frac{I_{k+1}(\alpha_{mk})}{I_{k-1}(\alpha_{mk})} = \frac{f+\lambda_{mk}}{f-\lambda_{mk}}. \tag{4.10}$$

Since the left-hand side of (4.10) is <1 and the azimuthal wavenumber $k>0$, it follows that solutions for the frequency λ_{mk}, if they exist at all, must have the property that λ_{mk} is negative. This conclusion is consistent with the general property 5 of §3. Hence class II oscillations are waves travelling around the cylinder in the same direction as the mean rotation.

For each m and k there exists at most one mode with frequency λ_{mk}, which is obtained by considering the intersection of the two curves $y_1(\lambda_{mk})$ and $y_2(\lambda_{mk})$:

$$y_1(\lambda_{mk}) = \frac{f+|\lambda_{mk}|}{f-|\lambda_{mk}|} \tag{4.11}$$

$$y_2(\lambda_{mk}) = \frac{I_{k-1}(\alpha_{mk})}{I_{k+1}(\alpha_{mk})} \tag{4.12}$$

$(k>0)$.

It is appropriate to consider separately the cases $f^2<N^2$ and $f^2>N^2$. For $f^2<N^2$, figure 1 gives a sketch of the curves $y_1(\lambda_{mk})$ and $y_2(\lambda_{mk})$ in the range $0<|\lambda_{mk}|<f$. There are two possibilities shown for $y_2(\lambda_{mk})$; the solid curve labelled (*a*) which intersects $y_2(\lambda_{mk})$ and thus produces a solution λ_{mk} to (4.9) and (4.10), and the dotted curve labelled (*b*) which does not. Hence the necessary and sufficient condition for the existence of an eigenvalue is

$$\lim_{\lambda_{mk}\to f}\frac{y_1(\lambda_{mk})}{y_2(\lambda_{mk})} \geqslant 1. \tag{4.13}$$

Since $|\lambda_{mk}|\to f$ corresponds to $\alpha_{mk}\to 0$, the limiting behaviour of $y_2(\lambda_{mk})$ can be determined using the power series expansion for the modified Bessel function. Using (4.11) and (4.12) the inequality (4.13) implies that a mode of frequency λ_{mk} exists for those wavenumbers m and k that satisfy the inequality

$$\frac{k(k+1)\,h^2}{(m\pi a)^2+k(k+1)\,h^2} \leqslant \frac{f^2}{N^2}. \tag{4.14}$$

We note that if there exist integers m and k such that

$$\frac{k(k+1)\,h^2}{(m\pi a)^2+k(k+1)\,h^2} = \frac{f^2}{N^2}, \tag{4.15}$$

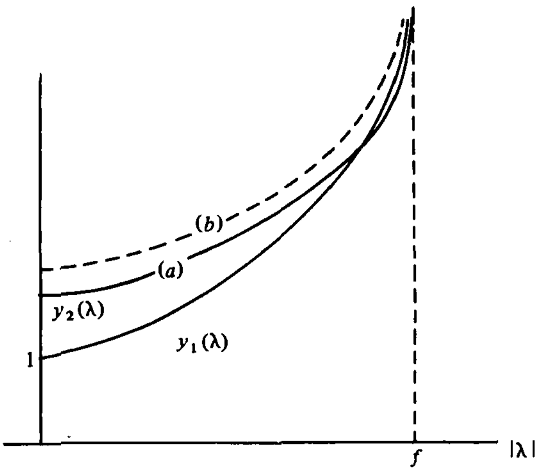

FIGURE 1. The possible intersection of the curves $y_1(\lambda)$ and $y_2(\lambda)$ in the case $f^2 < N^2$.

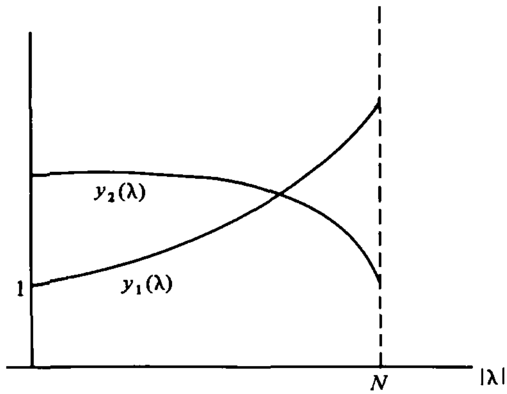

FIGURE 2. The intersection of the curves $y_1(\lambda)$ and $y_2(\lambda)$ in the case $N^2 < f^2$.

then the corresponding frequency $\lambda_{mk} = -f$. This result is in agreement with that obtained for the transition mode described in property 6 of §3. For the cylinder, equation (3.25) has solutions

$$g_k = \cos(m\pi z/h),$$

for those values of f^2/N^2 that satisfy (4.15). Hence the eigenfunction of a transition mode in the cylinder is

$$\Phi_{mk} = r^k \cos(m\pi z/h)\, e^{ik\theta},$$

with

$$\lambda_{mk} = -f.$$

In the case $f^2 > N^2$, figure 2 indicates the curves $y_1(\lambda_{mk})$ and $y_2(\lambda_{mk})$ in the range $0 < |\lambda_{mk}| < N$. Since $|\lambda_{mk}| \to N$ corresponds to $\alpha_{mk} \to 0$, an asymptotic expansion for the modified Bessel function with large argument is used to determine the behaviour of $y_2(\lambda_{mk})$. We find [see figure 2] that, for all wavenumbers m and k, there exists a unique point of intersection of the two curves, giving an eigenvalue $-|\lambda_{mk}|$ that

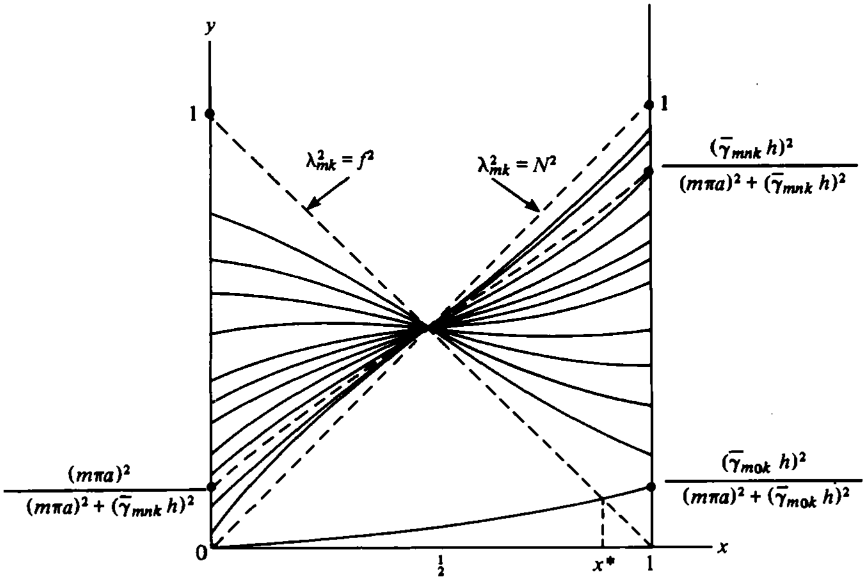

FIGURE 3. The frequency spectrum for fixed positive wavenumbers m and k and variable N^2/f^2; $x = N^2/(N^2+f^2)$, $y = \lambda^2_{mk}/(N^2+f^2)$.

satisfies (4.9) and (4.10). The asymptotic expression for λ_{mk} as it tends to $-N$ is given by

$$\lambda_{mk} \sim -N\left[\frac{(kh)^2}{(kh)^2+(m\pi a)^2}\right]^{\frac{1}{2}} \quad (m > 0,\ k > 0). \tag{4.16}$$

These solutions correspond to those obtained by Krauss (1966), who studied internal waves in rectangular geometry. Class II waves may be called internal Kelvin waves because they have a structure similar to that of Kelvin edge waves. Finally we observe that in this parameter range there is *no* transition mode $\lambda_{mk} = -N$ between the waves of class I and class II. This result is, of course, consistent with property 7 of §3 as applied to cylindrical geometry.

In the special case $f = N$, non-trivial solutions of class II, i.e. solutions to (4.9), require

$$\alpha^2_{mk} = \left(\frac{m\pi a}{n}\right)^2. \tag{4.17}$$

Substitution of (4.17) into (4.10) gives the explicit expression for the frequency of the class II wave,

$$\lambda_{mk} = -f\left[\frac{I_k(m\pi a/h)}{I'_k(m\pi a/h)}\right]\frac{kh}{m\pi a}. \tag{4.18}$$

Further consideration of this special case, including the mode at frequency $f = N$, is made in §5.

The behaviour of the frequency spectrum for both classes of internal waves, for fixed positive wavenumbers m and k and variable N^2/f^2, is illustrated in figure 3; the abscissa is $x = N^2/(N^2+f^2)$, $0 \leqslant x \leqslant 1$, and the ordinate is $y = \lambda^2_{mk}/(N^2+f^2)$, $0 < y < 1$. This transformation puts rotation (stratification) dominated behaviour at the left-hand (right-hand) side of the figure. Property 3 (§3) requires that all frequencies lie below the lines $y = x$ for $x > \frac{1}{2}$ and $y = 1-x$ for $x < \frac{1}{2}$. Each curve in the 'butterfly' represents a solution of class I to equations (4.5) and (4.6), and the single curve that lies below both $y = x$ and $y = 1-x$ represents the class II solution to equations (4.8) and (4.9). Of course, this curve arises for the cases $\lambda_{mk} < 0$ only.

The following argument can be used to indicate the shape of the curves corresponding to class I waves. Equation (4.6) can be written in the (x, y) variables as

$$y = x\left(\frac{\gamma^2_{mnk}h^2-(m\pi a)^2}{\gamma^2_{mnk}h^2+(m\pi a)^2}\right)+\frac{(m\pi a)^2}{\gamma^2_{mnk}h^2+(m\pi a)^2}. \tag{4.19}$$

When $x = 1$ (i.e. $f = 0$) the radial wavenumber $\overline{\gamma}_{mnk}/a$ is given via the nth positive zero of J'_k, and, except for the smallest root γ_{m0k}, the nth root γ_{mnk} varies little from $\overline{\gamma}_{mnk}$ as x decreases to zero. Consequently the nth curve in figure 3 lies close to the straight line given by substituting $\overline{\gamma}_{mnk}$ into (4.8). This line is shown dashed in figure 3, as are the boundary lines $\lambda^2_{mk} = f^2$ and $\lambda^2_{mk} = N^2$. If $k \neq 0$ and $\lambda < 0$, it can be shown that the nth curve is above the line for $0 \leqslant x < \frac{1}{2}$ and below the line for $\frac{1}{2} < x < 1$; this situation is indicated by the solid curve in figure 3. If $k \neq 0$ and $\lambda > 0$, the inequalities are reversed, while if $k = 0$ the nth curve coincides with the straight line.

The curve that corresponds to the class II wave is determined by rewriting (4.9) in (x, y) variables. From (4.15) it follows that in the region close to $x = 0$ ($N = 0$)

$$y \sim \frac{x(kh)^2}{(kh)^2+(m\pi a)^2}.$$

Thus, from (4.14), this solution may be regarded as a constituent of the steady mode in the limit of a homogeneous fluid with vanishingly small stratification (see Allen 1971 for a discussion of this limit). The curve for the class II mode intersects the line $y = 1-x$ at

$$x = \frac{(m\pi a^2+k(k+1)h^2)}{(m\pi a)^2+2k(k+1)h^2} \equiv x^*.$$

For $x > x^*$ there is no class II mode; however, the curve continues, as shown on figure 3, above the line $y = 1-x$. In this region the curve represents the class I mode whose radial wavenumber is given by the smallest solution of (4.5) and (4.6). This root approaches the smallest zero γ_{m0k} of J'_k as x approaches 1.

Completeness of the set of modal solutions (4.4)–(4.9) for k non-zero is an open question. However for any m and non-zero k, exactly one class I mode with $x > 0$ can be found with a value of γ_{mnk} lying between successive zeros of J_k. The same is true for $\lambda < 0$ if x is close enough to 1. Further, a class II mode can always be found for smaller x when a root does not occur between the first two zeros of J_k. This enumeration strongly suggests that the eigenfunction set is complete.

4.2. *The sphere*

Another geometry for which equations (2.21) and (2.22) can be separated is a sphere, which for simplicity we choose to be the unit sphere $r^2+z^2 = 1$. Solutions can be obtained by modifications of the procedure for a homogeneous fluid (Greenspan 1968). We seek solutions to equation (4.1) with boundary condition

$$r\frac{\partial\Phi}{\partial r}+\frac{f}{i\lambda}\frac{\partial\Phi}{\partial\theta}+\left(\frac{\lambda^2-f^2}{\lambda^2-N^2}\right)z\frac{\partial\Phi}{\partial z} = 0. \tag{4.20}$$

	Mode type		
	I	II ($N^2 < f^2$)	II ($N^2 > f^2$)
z	$\mu\xi\xi_s^{-1}$ ($\xi \leqslant \xi_s$)	$\mu\xi\xi_s^{-1}$ ($\xi \leqslant \xi_s$)	$\mu\xi\xi_s^{-1}$ ($\xi > \xi_s$)
r	$(1-\mu^2)^{\frac{1}{2}}(\xi_s^2 Q^{-2}-\xi^2)^{\frac{1}{2}}$	$(1-\mu^2)^{\frac{1}{2}}(\xi_s^2 Q^{-2}+\xi^2)^{\frac{1}{2}}$	$(1-\mu^2)^{\frac{1}{2}}(\xi^2-\xi_s^2 Q^{-2})^{\frac{1}{2}}$
Φ (n integral)	$P_m^k(\xi\xi_s^{-1}Q)\,P_m^k(\mu)\,e^{ik\theta}$	$P_m^k(i\xi\xi_s^{-1}Q)\,P_m^k(\mu)\,e^{ik\theta}$	$P_m^k(\xi\xi_s^{-1}Q)\,P_m^k(\mu)\,e^{ik\theta}$ ($0 < k < m$)
Eigenvalue equations	$fk\lambda^{-1}P_m^k(Q) = P_m^{k\prime}(Q)Q^{-1}(1-Q^2)$ ($0 < Q < 1$)	$fk\lambda^{-1}P_m^k(iQ) = -P_m^{k\prime}(iQ)Q^{-1}(1+Q^2)$ ($Q > 0$)	$fk\lambda^{-1}P_m^k(Q) = P_m^{k\prime}(Q)Q^{-1}(1-Q^2)$ ($Q > 1$)

TABLE 1. Separable modal solutions in a rotating stratified sphere;

$$Q \equiv \left|\frac{N^2-\lambda^2}{N^2-f^2}\right|^{\frac{1}{2}}, \quad \xi_s \equiv \left|\frac{N^2-\lambda^2}{f^2-\lambda^2}\right|^{\frac{1}{2}}.$$

Separable modal solutions are found by a transformation of the co-ordinate system (r,z) into an oblate spheroidal co-ordinate system (ξ,μ) introduced by Bryan (1889).

The solutions for the eigenfunctions and the eigenvalue equations for the class I and class II modes are given in table 1. We find that results are similar in many respects to those obtained in the cylinder. The class I modes are directly analogous to the modes that exist in a homogeneous rotating sphere. For instance, for $k > 0$ and both N^2 and f^2 non-zero, it can be shown that there is at most one more root of the class I eigenvalue equation than the number that exist for $N^2 = 0$, and no more roots than occur for $f^2 = 0$, just as for (4.5) and (4.6) in a cylinder.

To examine the class II modes ($\lambda^2 < \min(f^2, N^2)$), it is useful to consider separately the cases $f^2 < N^2$ and $f^2 > N^2$, as indicated in table 1. In both cases the eigenvalue equations permit at most one solution for k positive and λ negative, and none for λ positive (as predicted by property 5, § 3).

In the cylinder we showed that for $f^2 < N^2$ it is possible for transition modes between waves of class I and class II to exist. In this parameter range analogous results concerning the existence of transition modes are valid in the sphere. From property 6 (§ 3) the eigenfunction for the mode with frequency $\lambda_{mk} = -f$ is obtained via equation (3.30). In the case of a sphere $r^2+z^2 = 1$, (3.26) becomes

$$(1-z^2)\,g_k'' - 2z(k+1)\,g_k' + k(k+1)\left(\frac{N^2-f^2}{f^2}\right)g_k = 0. \tag{4.21}$$

Equation (4.21) is the associated Legendre equation and the solutions bounded at the poles are

$$g_k = P_m^k(z), \tag{4.22}$$

with

$$\frac{f^2}{N^2} = \frac{k(k+1)}{m(m+2k+1)+k(k+1)}. \tag{4.23}$$

Hence from (3.22) $\Phi_{mk} = P_m^k(z)\,r^k e^{ik\theta}$, with $\lambda_{mk} = -f$.

In the case $f^2 > N^2$ there is a significant difference between the geometries of the cylinder and the sphere. In the cylinder there is no transition mode; all the class II modes must become a constituent of the steady mode as $N^2 \to 0$. In contrast with this result, it can be shown, after considerable algebra, that in a sphere those class II modes with $(n-k)$ even cross over to become class I modes as N^2 decreases from f^2. From property 7 (§3) the eigenfunctions corresponding to $\lambda = -N$ are obtained via equation (3.41). In the case of a sphere, this equation becomes

$$(1-r^2)\left(h_k'' + \frac{1}{r}h_k' - \frac{k^2}{r^2}h_k\right) - rh_k' + \frac{kf}{N}h_k = 0. \tag{4.24}$$

This is a form of the standard hypergeometric equation, and to obtain polynomial solutions which remain bounded at $r = 0$ and $r = 1$ the values of f/N must be restricted to

$$kf/N = (2m+k+\tfrac{1}{2})^2 - (k^2+\tfrac{1}{4}), \tag{4.25}$$

with

$$h_k(r) = r^k F(-m, k+m+\tfrac{1}{2}; k+1; r^2), \tag{4.26}$$

where m is a positive integer (Erdelyi 1953). Using transformation identities (Abramowitz & Stegun 1965) for the polynomials in (4.26), we obtain from (3.32) the following expression for the eigenfunction

$$\Phi_{mk} = P^k_{2m+k}((1-r^2)^{\frac{1}{2}})\, e^{ik\theta} \tag{4.27}$$

with $\lambda_{mk} = -N$. We note that although the stratification N^2 at which transition occurs is always small (from (4.25) we conclude that $N < f/5$), the main point is that not all the oscillatory modes in a homogeneous rotating sphere are funnelled into the oscillations $\lambda^2 = f^2$ as N^2 increases from 0 to f^2. This difference between the cylinder and the sphere in spectral dependence on N^2/f^2 is a consequence of the fact that, in a rotating stratified sphere, the dependence of the motion on the co-ordinates r and z is closely related. However, in a cylinder the dependences are inherently distinct except for axisymmetric motions.

In the special case $f^2 = N^2$, the treatment of class II modes in the sphere is analogous to that of the cylinder. The equation (2.21) and boundary condition (2.22) reduce to

$$\nabla^2\Phi = 0 \tag{4.28}$$

with

$$\frac{\partial\Phi}{\partial\rho} + \frac{f}{i\lambda\rho}\frac{\partial\Phi}{\partial\theta} = 0 \quad \text{on} \quad \rho^2 = r^2+z^2 = 1. \tag{4.29}$$

Solutions of (4.28) and (4.29) can be written explicitly in spherical co-ordinates (ρ, ϕ, θ) as

$$\Phi_{mk} = e^{ik\theta}\rho^m P^k_m(\cos\phi), \quad \lambda_{mk} = -\frac{fk}{m}, \quad 0 < k < m. \tag{4.30}$$

Further discussion of the case $f^2 = N^2$ is contained in §5.

5. Oscillatory modes for $f^2 = N^2$

We next consider the time-dependent modes when the two natural frequencies of oscillation in the fluid are identical. For this case, results may be found for more general container geometries than only for those considered in the previous section.

Class I solutions: $\lambda^2 = f^2$

For $\lambda^2 = f^2$ the interval of possible frequencies for class I oscillations reduces to a single value. For the cylinder of § 4, figure 3 illustrates pointedly how these modes are focused into the natural frequency. To determine the structure of this mode in an arbitrary container, note that property 1 of § 3 does not forbid motion with vanishing pressure gradient in the case $\lambda^2 = f^2 = N^2$. In fact, (3.10) implies that the pressure gradient must be zero in this case. It follows from (2.17) and (2.19) that the velocity and temperature fields of the $\lambda = f$ mode, using cylindrical co-ordinates, are related by

$$v = iu, \quad T = \frac{i}{f} w, \tag{5.1}$$

where

$$\mathbf{Q}(\mathbf{r}) = (u, v, w) \quad \text{and} \quad \mathbf{r} = (r, \theta, z).$$

Thus, specification of this oscillation in any container requires determination from initial conditions of two complex functions u and w, which are constrained only to satisfy (2.18) and (2.20).

We note that the focusing of the class I modes when $f^2 = N^2$ might be conjectured to lead to 'resonant' modes with time behaviour such as $te^{\pm ift}$. If any such form of resonance is assumed to occur, it can be shown that a contradiction arises. As a particular example, suppose solutions are sought with Φ oscillatory but $\mathbf{Q}$ and T growing linearly with time:

$$\begin{pmatrix} \mathbf{Q} \\ T \\ \Phi \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_0 \\ T_0 \\ \Phi_0 \end{pmatrix} e^{ift} + \begin{pmatrix} \mathbf{Q}_1 \\ T_1 \\ 0 \end{pmatrix} te^{ift}. \tag{5.2}$$

Using (5.2) in (2.5) and (2.7)–(2.9) and separating te^{ift} and e^{ift} terms gives

$$if\mathbf{Q}_1 + f\hat{\mathbf{k}} \times \mathbf{Q}_1 - f^2 T_1 \hat{\mathbf{k}} = 0, \tag{5.3}$$

$$if\mathbf{Q}_0 + f\hat{\mathbf{k}} \times \mathbf{Q}_0 - f^2 T_0 \hat{\mathbf{k}} + \nabla\Phi_0 + \mathbf{Q}_1 = 0, \tag{5.4}$$

$$ifT_1 + w_1 = 0, \tag{5.5}$$

$$ifT_0 + w_0 + T_1 = 0, \tag{5.6}$$

along with (2.18) and (2.20) for both $\mathbf{Q}_0$ and $\mathbf{Q}_1$. Eliminating T_1 from (5.3) using (5.5), and scalar-multiplying the conjugate of the result with $\mathbf{Q}_0$ gives

$$-if\mathbf{Q}_1^* \cdot \mathbf{Q}_0 + f\hat{\mathbf{k}} \times \mathbf{Q}_1^* \cdot \mathbf{Q}_0 + ifw_1^* w_0 = 0. \tag{5.7}$$

Then after eliminating T_0 from (5.4) with (5.6) and finding the scalar product with $\mathbf{Q}_1^*$ we obtain

$$if\mathbf{Q}_1^* \cdot \mathbf{Q}_0 + f\hat{\mathbf{k}} \times \mathbf{Q}_0 \cdot \mathbf{Q}_1^* - ifw_0 w_1^* + |\mathbf{Q}_1|^2 - ifT_1 w_1^* + \nabla\Phi_0 \cdot \mathbf{Q}_1^* = 0. \tag{5.8}$$

Adding (5.7) and (5.8), using (5.5) and the continuity equation, integrating over the container, and applying the boundary condition gives

$$\iiint_V \{|\mathbf{Q}_1|^2 + f^2 |T_1|^2\}\, dv = 0. \tag{5.9}$$

It follows from (5.9) that no resonance mode of the form (5.2) can occur in a finite container.

Class II solutions: $0 < \lambda^2 < f^2$

In contrast to the class I modes, the structure and spectrum of the class II modes do not undergo a significant change when $f^2 \to N^2$. For instance, figure 3 suggests that the frequency of the class II mode in the cylinder varies smoothly as f^2 increases or decreases through N^2. Further, it is noted in property 5 of § 3 that a pressure gradient must exist for class II modes when $N^2 = f^2$. The principal simplification in the explicit calculation of these modes for $f^2 = N^2$ in an arbitrary container is that the frequency λ disappears from the eigenfunction equation. That is, from (2.21) it follows that the pressure distribution is harmonic,

$$\nabla^2\Phi = 0. \tag{5.10}$$

Further, the boundary condition (2.22) reduces to

$$i\lambda\hat{\mathbf{n}}.\nabla\Phi + f\hat{\mathbf{k}}\times\hat{\mathbf{n}}.\nabla\Phi = 0 \quad \text{on} \quad B. \tag{5.11}$$

Solutions for the modal structure and frequencies in a cylinder are given by (4.7), (4.17) and (4.18), while those in a sphere are given by (4.30). The collection of class II oscillations in a sphere exhibits the property that each allowed frequency, i.e. any rational number between 0 and $-f$, has an infinite number of polynomial eigenfunctions associated with it.

As for the class I modes when $f^2 = N^2$, it is readily possible to investigate the class II modes in more general geometries than when $f^2 \neq N^2$. We shall confine attention to containers for which the z axis is an axis of symmetry, although it is evidently possible to extend some of our results to more general containers. We denote by θ the periodic co-ordinate which increases in the direction of $\hat{\mathbf{k}} \times \hat{\mathbf{n}}$.

We have shown in property 5 of § 3 that class II modes propagate only in the same direction as the mean rotation, i.e. in the direction of increasing θ. This result is valid for all f^2 and N^2. In the case $f^2 = N^2$, (3.19) reduces to

$$\iiint_V |\nabla\Phi|^2\,dv = -\frac{fi}{\lambda}\iint_B \Phi^*(\hat{\mathbf{k}}\times\hat{\mathbf{n}}).\nabla\Phi\,d\Sigma. \tag{5.12}$$

We note that (5.12) could be the basis for a variational principle for the determination of λ.

We provide a further example of the solution of (5.10) and (5.11) by taking the boundary B to be a prolate spheroid. The interior of the spheroid can be expressed as

$$(r,\theta,z) = (A\sinh\xi\sin\phi, \theta, A\cosh\xi\cos\phi), \quad 0 \leqslant \xi < c, \quad 0 \leqslant \phi \leqslant \pi, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{5.13}$$

and $\xi = c$ is the boundary B. We choose $A\sinh c = 1$, so that in cylindrical co-ordinates B is given explicitly by

$$r^2 + (1-d^{-2})z^2 = 1, \tag{5.14}$$

where $d = \cosh c$. Since (5.10) separates in prolate spheroidal co-ordinates, the bounded solutions $\Phi = e^{ik\theta}\chi$ which are of interest here are

$$\chi(\xi,\phi) = P_m^k(\cosh\xi)\,P_m^k(\cos\phi), \quad 0 < k < m. \tag{5.15}$$

With $\hat{\mathbf{n}} = \hat{\boldsymbol{\xi}}$ and equations (5.13) and (5.15), it can be shown that (5.11) reduces to

$$i\lambda(d^2-1)\,P_m^{k\prime}(d) + ikfdP_m^k(d) = 0. \tag{5.16}$$

Using a standard recurrence relation for Legendre functions (Gradshteyn & Ryzhik 1965), (5.16) implies that frequencies of oscillation λ for class II modes in the prolate spheroid satisfy

$$(\lambda m + fk)\, dP_m^k(d) - \lambda(m+k)\, P_{m-1}^k(d) = 0, \quad 0 < k < m. \tag{5.17}$$

Note there is a one-to-one correspondence *via* (4.30), (5.15) and (5.17) between modes in a sphere and in a prolate spheroid.

From (5.14) it follows that the limit $d \to \infty$ corresponds to the situation when B is the unit sphere. Further, when $d \to \infty$ the term dP_m^k in (5.17) dominates P_{m-1}^k. Thus, for the sphere, we can recover from (5.17) the result (4.30). For the case of a nearly spherical spheroid, i.e. for large finite d, the frequencies of oscillation can be obtained from the asymptotic behaviour of (5.17). Employing the formula (Gradshteyn & Ryzhik 1965)

$$P_m^k(d) \sim \frac{(2m)!\,(-1)^{k/2}}{(m-k)!\,m!\,2^m} d^m + O(d^{m-2}), \quad d \to \infty, \tag{5.18}$$

and expanding the frequency for any allowed n and k in an asymptotic series,

$$\lambda \sim \lambda^{(0)} + d^{-2}\lambda^{(1)} + \ldots, \quad \lambda^{(0)} = -fk/m, \tag{5.19}$$

we find from the $O(d^{m-1})$ terms in (5.17) that

$$\lambda^{(1)} = \frac{m^2-k^2}{m(2m-1)} \lambda^{(0)}, \quad 0 < k < m. \tag{5.20}$$

Thus, for any oscillation with frequency $\lambda^{(0)}$ in a sphere, (5.20) provides an $O(d^{-2})$ frequency correction when the sphere is deformed slightly into the prolate spheroid given by (5.14).

Investigation of containers for which Laplace's equation does not separate when the boundary B is a constant-co-ordinate surface may be made by various approximation methods. One of these would apply if the boundary differed slightly from one for which the solutions were easily obtained. For example, suppose B is a small perturbation from the unit sphere which is axisymmetric but is otherwise arbitrary. We let the exterior normal $\mathbf{n}$ be given by

$$\mathbf{n} = \hat{\boldsymbol{\rho}} + \epsilon\gamma(\phi)\hat{\boldsymbol{\phi}}, \tag{5.21}$$

where $\epsilon \ll 1$. (Since we have continued to consider only axisymmetric boundaries, γ is independent of θ, and there is no perturbation in the $\hat{\boldsymbol{\theta}}$ direction.)

To obtain approximations for the modal structures and frequencies, we now expand

$$\left.\begin{aligned} \Phi(\mathbf{r}, \epsilon) &= \Phi^{(0)}(\mathbf{r}) + \epsilon\Phi^{(1)}(\mathbf{r}) + \ldots, \\ \lambda(\epsilon) &= \lambda^{(0)} + \epsilon\lambda^{(1)} + \ldots, \end{aligned}\right\} \tag{5.22}$$

and apply (5.11) on $\rho = 1 - \epsilon D_\phi^{-1}(\gamma) + \ldots$. The lowest-order solutions for $\Phi^{(0)}$ and $\lambda^{(0)}$ are just those for the spherical boundary, given in (4.30) for any positive integers k and m $(0 < k < m)$. For any fixed values of k and m, the first-order corrections $\Phi^{(1)}$ and $\lambda^{(1)}$ for this mode satisfy the following equations, obtained by using (5.22) in (5.10) and (5.11):

$$\nabla^2\Phi^{(1)} = 0, \tag{5.23}$$

$$i\lambda^{(0)}\Phi_\rho^{(1)} + f\Phi_\theta^{(1)} = -\{i\lambda^{(1)}\Phi_\rho^{(0)} + \gamma(\phi)\,[i\lambda^{(0)}\Phi_\phi^{(0)} + f\cot\phi\,\Phi_\theta^{(0)}]\} \quad \text{on} \quad \rho = 1. \tag{5.24}$$

The appropriate solution of (5.23) may be chosen as

$$\Phi^{(1)} = e^{ik\theta} \sum_{j=k}^{\infty} C_j \rho^j P_j^k(\cos\phi), \tag{5.25}$$

where C_m is arbitrary and the C_j for $j \neq m$ remain to be determined. Introducing (4.30) and (5.25) into (5.24) and using the same recurrence relation which simplified (5.16) produces the result

$$\sum_{j=k}^{\infty} \frac{fk}{m}(m-j)\, C_j P_j^k(\cos\phi) = -\lambda_1 m P_m^k(\cos\phi) - \left(\frac{fk}{m}\right)\gamma(\phi)\frac{(m+k)}{\sin\phi}P_{m-1}^k(\cos\phi). \tag{5.26}$$

Orthogonality of the Legendre functions, along with the solvability requirement, implies from (5.26) that

$$C_j = -\frac{m+k}{m-j}D_j, \quad j \neq m, \tag{5.27}$$

and

$$\lambda^{(1)} = \frac{m+k}{m}\lambda^{(0)} D_m, \tag{5.28}$$

where

$$D_j = \frac{\int_0^\pi \gamma(\phi)\, P_{m-1}^k(\cos\phi)\, P_j^k(\cos\phi)\, d\phi}{\int_0^\pi [P_j^k(\cos\phi)]^2 \sin\phi\, d\phi}. \tag{5.29}$$

Thus equations (5.28) and (5.29) provide the $O(\epsilon)$ frequency correction to any oscillation with frequency $\lambda^{(0)}$ when the boundary of the sphere undergoes an axisymmetric perturbation satisfying (5.21). Equations (5.25), (5.27) and (5.29) represent the corresponding change in modal structure.

The arbitrariness of the boundary perturbation means that the results (5.28) and (5.29) considerably generalize (5.20), for which the deformed surface must be prolate-spheroidal. Indeed, we can show that (5.28) and (5.29) must reduce to (5.20) for this case as follows. Determination of the exterior normal to (5.14) in spherical co-ordinates and for large d leads to the identifications $\epsilon = d^{-2}$ and

$$\gamma(\phi) = \cos\phi \sin\phi, \tag{5.30}$$

in (5.21). Using (5.30) in (5.29) along with the formula (Gradshteyn & Ryzhik 1965)

$$(2m-1)\cos\phi P_{m-1}^k(\cos\phi) = (m-k)\, P_m^k(\cos\phi) + (m-1+k)\, P_m^k(\cos\phi) \tag{5.31}$$

yields $D_m = (m-k)/(2m-1)$. Therefore, equation (5.20) is confirmed as a special case of (5.28). We note finally that a small non-axisymmetric boundary perturbation is analytically more cumbersome. However, the rather laborious perturbation analysis can be performed even in the case of a non-axisymmetric deformation of the boundary. Unlike (5.25), corrections for an $O(1)$ mode corresponding to a single azimuthal wavenumber k cannot in general be obtained independently. Rather, the corrections can be determined to the composite mode

$$\Phi_m^{(0)} = \sum_{k=1}^{m} b_k e^{ik\theta} \rho^m P_m^k(\cos\phi),$$

as functions of the coefficients b_k.

6. The initial-value problem in the cylinder

We are now in the position to give the formal solution to the initial-value problem in the cylinder. We seek the solution to the problem given by equations (2.7)–(2.9) with boundary conditions (2.5) and initial condition (2.6), when the container surface B is a cylinder of radius a and height h. We write the solution for the total pressure field as a superposition of the steady field and time-dependent normal modes

$$P(\mathbf{r},t) = P_g(\mathbf{r}) + \mathscr{R} \sum_{\substack{m>0 \\ n>0 \\ k\geqslant 0}} \exp[i\lambda_{mn}t]\, a_{mnk}\, \Phi_{mnk}(\mathbf{r}). \tag{6.1}$$

We denote class II modes in the summation (6.1) by coefficients a_{mk}.

The geostrophic pressure P_g satisfies (2.11), (2.12) and (2.13) in the case that the boundary surface is a cylinder

$$\nabla_H^2 P_g + \frac{f^2}{N^2}\frac{\partial^2 P_g}{\partial z^2} = f\hat{\mathbf{k}}.\nabla\times\mathbf{q}_0 + f^2\frac{\partial\tau_0}{\partial z} \tag{6.2}$$

with

$$\frac{\partial P_g}{\partial z} = N^2\tau_0 \quad \text{on} \quad z = 0, h \tag{6.3}$$

and

$$\int_0^{2\pi} \frac{\partial P_g}{\partial r}(a,z,\theta)\, a\, d\theta = f\int_0^{2\pi} \mathbf{q}_0(a,\theta,z)\,.\,\hat{\boldsymbol{\theta}} a\, d\theta \tag{6.4}$$

and where

$$P_g = \text{constant} \quad \text{on} \quad r = a \quad \text{for fixed } z.$$

We note that equation (6.2) represents the fact that the geostrophic part of the solution to an initial-value problem in a geostrophically free region is that unique geostrophic flow which has the same potential vorticity as that of the initial flow. It is also the case that on a horizontal boundary the temperature must equal its initial value (6.3), and on a non-horizontal boundary the 'potential circulation' must equal its initial value (6.4). The solution to (6.2)–(6.4) can be represented as a Fourier series in θ;

$$P_g = \sum_{k\geqslant 0} P^k(r,z)\, e^{ik\theta}$$

(for details see Howard & Siegmann 1969).

The eigenfunctions $\Phi_{mnk}(\mathbf{r})$ and the frequencies λ_{mnk} of the oscillatory modes in the cylinder have been determined explicitly in §4. To complete the initial-value problem, it is therefore necessary to compute the coefficients a_{mnk}. To do this it is desirable to consider separately the two cases $f^2 \neq N^2$ and $f^2 = N^2$. In the first case there exists a triply infinite family of class I modes, whereas in the second case these modes degenerate to a single mode with $\lambda^2 = f^2$.

Case 1: $f^2 \neq N^2$

In §3, property 4, we proved that eigenfunctions $(\mathbf{Q}_{(\lambda)}, T_{(\lambda)})$ and $(\mathbf{Q}_{(\mu)}, T_{(\mu)})$ corresponding to distinct eigenvalues λ and μ are orthogonal. We will use this property to compute the coefficients a_{mnk}. First the velocity and temperature can be represented in terms of normal modes by applying the appropriate differential operators to (6.1). The only possible difficulty in this process is the term-by-term differentiation with respect to r. This procedure is clearly permissible for the $k = 0$ series and may be justified in the usual way if $k \neq 0$. We write

$$\mathbf{q}(r,t) = \frac{1}{f}\hat{\mathbf{k}} \times \nabla P_g + \mathscr{R} \sum_{\substack{m>0\\n>0\\k\geqslant 0}} \exp\left[i\lambda_{mnk}t\right] a_{mnk}\, \mathbf{Q}_{mnk}(\mathbf{r}), \tag{6.5}$$

and

$$\tau(\mathbf{r},t) = \frac{1}{N^2}\frac{\partial P_g}{\partial z} + \mathscr{R} \sum_{\substack{m>0\\n>0\\k\geqslant 0}} \exp\left[i\lambda_{mnk}t\right] a_{mnk} T_{mnk}(\mathbf{r}), \tag{6.6}$$

where

$$Q_{mnk}(\mathbf{r}) = \frac{i\lambda_{mnk}}{\lambda_{mnk}^2 - f^2}\left[\nabla\Phi_{mnk} + \left(\frac{N^2-f^2}{\lambda_{mnk}^2 - N^2}\right)\frac{\partial \Phi_{mnk}}{\partial z}\hat{\mathbf{k}} - \frac{f\hat{\mathbf{k}} \times \nabla\Phi_{mnk}}{i\lambda_{mnk}}\right], \tag{6.7}$$

and

$$T_{mnk}(\mathbf{r}) = \frac{N^2}{N^2 - \lambda_{mnk}^2}\frac{\partial \Phi_{mnk}}{\partial z}. \tag{6.8}$$

We apply the initial condition (2.6) to (6.5) and (6.6) to obtain

$$\mathbf{Q}_0(\mathbf{r}) = \left(\mathbf{q}_0(\mathbf{r}) - \frac{1}{f}\hat{\mathbf{k}} \times \nabla P_g\right) = \mathscr{R} \sum_{\substack{m>0\\n>0\\k\geqslant 0}} a_{mnk}\, \mathbf{Q}_{mnk}(\mathbf{r}) \tag{6.9}$$

and

$$T_0(\mathbf{r}) = \left(\tau_0 - \frac{1}{N^2}\frac{\partial P_g}{\partial z}\right) = \mathscr{R} \sum_{\substack{m>0\\n>0\\k\geqslant 0}} a_{mnk} T_{mnk}(\mathbf{r}). \tag{6.10}$$

We therefore wish to express $\mathbf{Q}_0(\mathbf{r})$ and $T_0(\mathbf{r})$ as Fourier series in the functions $Q_{mnk}(\mathbf{r})$ and $T_{mnk}(\mathbf{r})$ given by (6.7) and (6.8). In §4 we computed the eigenfunctions $\Phi_{mnk}(\mathbf{r})$ for the pressure in the cylinder, which are given by (4.4) and (4.7). The eigenfunction sets for $\mathbf{Q}_0(\mathbf{r})$ and $T_0(\mathbf{r})$ are clearly complete in the θ and z variables. As we remarked in §4, the enumeration of the zeros of J_k and I_k strongly suggests, but does not prove, that there are sufficiently many eigenfunctions in the r-variable to complete the representation of $\mathbf{Q}_0(\mathbf{r})$ and $T_0(\mathbf{r})$.

We assume that the Fourier series (6.9) and (6.10) converge uniformly to $\mathbf{Q}_0(\mathbf{r})$ and $T_0(\mathbf{r})$ respectively (i.e. we assume that $\mathbf{Q}_0$ and T_0 are twice differentiable with respect to θ and z and sufficiently differentiable with respect to r). Term-by-term integration is then justified and the coefficients a_{mnk} can be computed using the orthogonality condition (3.11):

$$a_{mnk} = \frac{\iiint [\mathbf{Q}_0 \cdot \mathbf{Q}_{mnk}^* + N^2 T_0 T_{mnk}^*]\, dv}{\iiint [|\mathbf{Q}_{mnk}|^2 + N^2|T_{mnk}|^2]\, dv}. \tag{6.11}$$

A conceivable difficulty is the possibility that eigenvalues whose subscripts (m, n, k) are not identical may not be distinct. In the cylinder the eigenvalues are given by (4.5) and (4.6) or (4.8) and (4.9). Clearly, two eigenvalues with the same subscripts k and m but distinct subscript n must be distinct. It is possible that two eigenvalues could have distinct subscripts k and suitable distinct values of m and n that allow the two eigenvalues to be identical. In this case, however, the orthogonality of the set $\{e^{ik\theta}\}$ ensures that a_{mnk} is determined as in (6.11) without difficulty.

Case $2: f^2 = N^2$

In this case all the class I modes are focused into a single natural frequency. The structure of this oscillation is determined in §5; the pressure gradient vanishes and the frequency f mode consists of two arbitrary complex functions as given in (5.1). The class II modes remain a double infinite set. For the cylinder the eigenfunctions and eigenvalues are given explicitly in §4 by (4.7), (4.17) and (4.18).

The pressure, velocity and temperature written in terms of a superposition of the geostrophic mode, the class I mode and the class II modes have the forms

$$P(\mathbf{r},t) = P_g(\mathbf{r}) + \mathscr{R} \sum_{\substack{m>0\\k>0}} b_{mk}\,\Phi_{mk}(\mathbf{r})\exp[i\lambda_{mk}t], \tag{6.12}$$

$$\mathbf{q}(\mathbf{r},t) = \frac{1}{f}\hat{\mathbf{k}}\times\nabla P_g + \mathscr{R}\tilde{\mathbf{Q}}e^{ift} + \mathscr{R}\sum_{\substack{m>0\\k>0}} b_{mk}\,\mathbf{Q}_{mk}(\mathbf{r})\exp[i\lambda_{mk}t], \tag{6.13}$$

$$\tau(\mathbf{r},t) = \frac{1}{N^2}\frac{\partial P_g}{\partial z} + \mathscr{R}\frac{i\tilde{w}}{f}e^{ift} + \mathscr{R}\sum_{\substack{m>0\\k>0}} b_{mk}\,T_{mk}(\mathbf{r})\exp[i\lambda_{mk}t], \tag{6.14}$$

where

$$\tilde{\mathbf{Q}} = (\tilde{u}, i\tilde{u}, \tilde{w}), \quad \tilde{T} = i\tilde{w}/f, \tag{6.15}$$

and $\mathbf{Q}_{mk}$ and T_{mk} are given by (6.7) and (6.8) with $f^2 = N^2$.

We set (6.13) equal to the given initial velocity $\mathbf{q}_0(\mathbf{r})$. The azimuthal component of this expression is

$$v_0 - \frac{1}{f}\frac{\partial P_g}{\partial r} = \mathscr{R}i\tilde{u} + \mathscr{R}\sum_{\substack{m>0\\k>0}} b_{mk}\,v_{mk}(\mathbf{r}). \tag{6.16}$$

The boundary condition applied to the frequency f mode requires $\tilde{u}(a,\theta,z) = 0$. The components $v_{mk}(\mathbf{r})$ can be evaluated at $r = a$ using (4.7) and (6.7) with $f^2 = N^2$ to give

$$v_{mk}(a,\theta,z) = \left(\frac{m\pi}{fh}\right)e^{ik\theta}\cos\left(\frac{m\pi z}{h}\right)I_k'\left(\frac{m\pi a}{h}\right). \tag{6.17}$$

We determine the coefficients b_{mk} by evaluating (6.16) at $r = a$ and using the orthogonality of $\{e^{ik\theta}\}$ and $\{\cos m\pi z/h\}$ to give

$$b_{mk} = \frac{2f}{m\pi^2 I_k'(m\pi a/h)}\int_0^{2\pi} e^{-ik\theta}\int_0^h \left(v_0 - \frac{1}{f}\frac{\partial P_g}{\partial r}\right)\cos(m\pi z/h)\,dz\,d\theta. \tag{6.18}$$

To justify the term-by-term integration implicit in this procedure, we assume that $(v_0 - (1/f)\,\partial P_g/\partial r)_a$ is twice differentiable with respect to θ and z.

Once the coefficients b_{mk} are computed from (6.18), the unknown complex functions $\tilde{u}$ and $\tilde{w}$ are determined from, for example, the r and z components of (6.13) evaluated at $t = 0$. This procedure gives the velocity and temperature in real function notation as

$$\begin{aligned}
u(\mathbf{r},t) &= (u_0 - \bar{u}_{t=0})\cos ft + (v_0 - \bar{v}_{t=0})\sin ft + \bar{u}(\mathbf{r},t),\\
v(\mathbf{r},t) &= (v_0 - \bar{v}_{t=0})\cos ft - (u - \bar{u}_{t=0})\sin ft + \bar{v}(\mathbf{r},t),\\
w(\mathbf{r},t) &= (w_0 - \bar{w}_{t=0})\cos ft + f(\tau_0 - \bar{\tau}_{t=0})\sin ft + \bar{w}(\mathbf{r},t),\\
\tau(\mathbf{r},t) &= (\tau_0 - \bar{\tau}_{t=0})\cos ft - \frac{1}{f}(w_0 - \bar{w}_{t=0})\sin ft + \bar{\tau}(\mathbf{r},t),
\end{aligned}$$

where

$$\bar{\mathbf{q}}(\mathbf{r},t) = \frac{1}{f}\hat{\mathbf{k}}\times\nabla P_g + \mathscr{R}\sum_{\substack{m>0\\k>0}} b_{mk}\mathbf{Q}_{mk}(\mathbf{r})\exp[i\lambda_{mk}t]$$

and

$$\bar{\tau}(r,t) = \frac{1}{N^2}\frac{\partial P_g}{\partial z} + \mathscr{R}\sum_{\substack{m>0\\k>0}} b_{mk}T_{mk}(\mathbf{r})\exp[i\lambda_{mk}t].$$

It is clear that by construction this solution satisfies the boundary condition, $u(\mathbf{r},t) = 0$ at $r = a$, provided the initial radial velocity u_0 satisfies this condition. It can also be verified that the solution satisfies the conservation equation, again provided that the initial velocity $\mathbf{q}_0$ satisfies the conservation equation.

Examples

The coefficients a_{mnk} can be computed readily for certain simple examples of initial conditions. In the case of the following axisymmetric initial conditions,

$$\tau_0 = 0, \quad \mathbf{q}_0 = \hat{\boldsymbol{\theta}} J_1(\beta r/a)\cos\pi z/h, \tag{6.19}$$

where

$$J_1(\beta) = 0,$$

the geostrophic mode P_g is given by

$$P_g = \frac{f\beta^2}{\beta^2/a^2 + f^2\pi^2/N^2h^2} J_1(\beta r/a)\, e^{i\theta}\cos\pi z/h. \tag{6.20}$$

For the class I modes $\mathbf{Q}_{mnk}$ and T_{mnk} are computed from (6.7), (6.8) and (4.4). The integral in the numerator of the expression (6.11) for a_{mnk} can then be calculated with $\mathbf{Q}_0$ and T_0 determined from (6.19) and (6.20). It is a fairly simple procedure to show that the only non-zero coefficient a_{mnk} occurs when $k = 0$, $m = 1$, and the nth zero γ_{1nk} of J_1 is equal to β. For the class II modes, property 5 of § 3 shows that there are no modes with $k = 0$; hence, in this example, no class II modes are generated. Thus we conclude that an initial disturbance composed of a single axisymmetric mode will excite only the geostrophic mode and a single class I wave whose radial and vertical wavenumbers are exactly those of the initial disturbance.

In the case of a non-axisymmetric initial condition, the problem of computing the coefficients a_{mnk} becomes somewhat more complicated. We consider the example where

$$\tau_0 = 0, \quad \mathbf{q}_0 = \hat{\boldsymbol{\theta}} e^{i\theta}\cos(\pi z/h)\,(1/r)\int rJ_1\left(\frac{\beta r}{a}\right)dr, \tag{6.21}$$

where

$$J_1(\beta) = 0.$$

In this case the geostrophic mode is given by

$$P_g = \frac{-f}{\beta^2/a^2 + f^2\pi^2/N^2h^2} J_1\left(\frac{\beta r}{a}\right) e^{i\theta}\cos\left(\frac{\pi z}{h}\right). \tag{6.22}$$

Rather laborious computations of the integrals obtained by substituting (6.7), (6.8) and (4.4) together with (6.21) and (6.22) into (6.11) yield the following information.

The coefficients a_{mnk} are zero for $m \neq 1$, $k \neq 1$. However it is no longer the case that $a_{|n|}$ is zero for $\gamma_{|n|} \neq \beta$. (In fact, we observe that β is not a root of the transcendental equation (4.5) that defines $\gamma_{|n|}$.) An initial non-axisymmetric disturbance will excite modes of all radial wavenumbers. The dominant mode will be the mode whose radial wavenumber $\gamma_{|n|}$ is nearest to β. In the parameter ranges for which a class II mode exists, this mode will always be excited. The amplitude of the class II mode will be non-negligible relative to the amplitude of the class I modes.

In the special case $f^2 = N^2$, the results are analogous to those that hold for $f^2 \neq N^2$. An axisymmetric initial velocity will generate only the geostrophic mode and the single class I mode of frequency f. A non-axisymmetric initial disturbance will also excite the class II modes. For the example given by (6.21) the coefficients b_{mk} can be computed from (6.18). It is clear that $b_{mk} = 0$ if $m \neq 1$ and $k \neq 1$.

We have indicated that a non-axisymmetric initial disturbance excites the class II modes with non-negligible amplitude. It is of interest to consider the velocity field associated with a class II mode. We substitute the expression Φ_{mk} given by (4.7) into equations (2.16)–(2.19) to obtain the following expressions for the velocity components of the class II mode:

$$u_{mk}(r,\theta,z,t) = \frac{\alpha_{mk}}{2a(f^2-\lambda_{mk}^2)}\left[(f-|\lambda_{mk}|)\,I_{k-1}\left(\frac{\alpha_{mk}r}{a}\right)-(f+|\lambda_{mk}|)\,I_{k+1}\left(\frac{\alpha_{mk}r}{a}\right)\right] \times\left(\cos\frac{m\pi z}{h}\sin(k\theta-|\lambda_{mk}|t)\right), \tag{6.23}$$

$$v_{mk}(r,\theta,z,t) = \frac{\alpha_{mk}}{2a(f^2-\lambda_{mk}^2)}\left[(f-|\lambda_{mk}|)\,I_{k-1}\left(\frac{\alpha_{mk}r}{a}\right)+(f+|\lambda_{mk}|)\,I_{k+1}\left(\frac{\alpha_{mk}r}{a}\right)\right] \times\left(\cos\frac{m\pi z}{h}\cos(k\theta-|\lambda_{mk}|t)\right) \tag{6.24}$$

$$w_{mk}(r,\theta,z,t) = \frac{\lambda_{mk}}{N^2-\lambda_{mk}^2}\left[\frac{m\pi}{h}\,I_k\left(\frac{\alpha_{mk}r}{a}\right)\right]\left(\sin\frac{m\pi z}{h}\sin(k\theta-|\lambda_{mk}|t)\right) \tag{6.25}$$

for $k > 1$, $m > 1$ and $\alpha_{mk} \neq 0$.

Figure 4 illustrates the r-dependence of the velocity components given by (6.23)–(6.25). We note that u_{mk}, v_{mk} and w_{mk} are all zero at the centre of the cylinder $r = 0$, and of course u_{mk} is zero at the boundary $r = a$. The components v_{mk} and w_{mk} are at maxima at the boundary $r = a$; however, the radial component u_{mk} achieves its largest value in the interior. The turning point for u_{mk} as a function of r is at the point r_0 satisfying the equation

$$\frac{I'_{k-1}(\alpha_{mk}r_0/a)}{I'_{k+1}(\alpha_{mk}r_0/a)} = \frac{f+|\lambda_{mk}|}{f-|\lambda_{mk}|}, \tag{6.26}$$

with

$$\alpha_{mk}^2 = \left(\frac{m\pi a}{h}\right)^2\left(\frac{f^2-\lambda_{mk}^2}{N^2-\lambda_{mk}^2}\right).$$

As $\lambda_{mk}^2 \to f^2$ the quantity $\alpha_{mk} \to 0$. In this case we use the power-series expansion for the modified Bessel function (Relton 1946) to approximate the solution of (6.26) as

$$r_0^2 \sim \left(\frac{h}{m\pi}\right)^2\frac{N^2-f^2}{f^2}k(k-1). \tag{6.27}$$

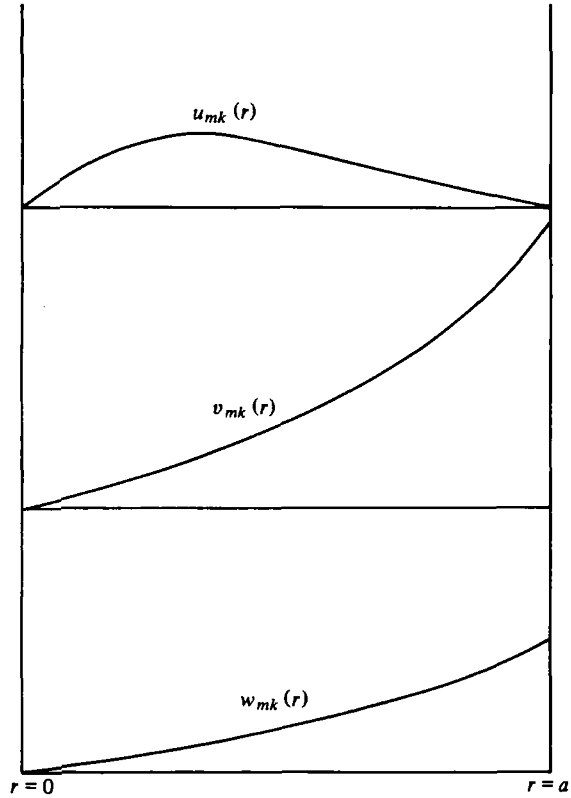

FIGURE 4. The r-dependent behaviour of the velocity components of a class II wave in a cylinder for fixed positive wavenumbers m and k.

The condition (4.14) for the existence of class II modes guarantees that $r_0^2 < a^2$. Clearly, the peak in the radial velocity component occurs close to the centre of the cylinder for those modes with low azimuthal wavenumber k. When $f^2 \ll N^2$, (6.28) can be further approximated by

$$r_0 \sim \frac{k}{m\pi}\frac{Nh}{f}.$$

We note that Nh/f is the Rossby radius of deformation, which for the ocean is typically about 60 km.

The above observation concerning class II modes can be compared with the situation for baroclinic Kelvin waves. These waves have been discussed by many authors, including Wang & Mooers (1976), Smith (1978) and Mysak (1980). They obtain long-wave approximations for waves trapped at a rectilinear boundary or channel, with the condition imposed of decay away from the boundary. Those solutions are similar to the class II waves of this present paper in that, for example, v_{mk} and w_{mk} decay away from the boundary. However, in the rectilinear trapped-wave solution, the radial component of velocity is identically zero, whereas in our case u_{mk} achieves its maximum in the interior.

We remark that viewing a baroclinic Kelvin wave purely as an edge wave phenomenon may be misleading. Rather, the internal Kelvin wave is a low-frequency (sub-inertial) natural mode of the rotating stratified fluid in a contained geometry.

It is a member, albeit a somewhat anomalous one, of the class of free modes of the system. The vertical and long-shore components of velocity are significant only at the boundary. However, the effects of the interval Kelvin wave are not completely confined to the edge, since the velocity component normal to the boundary peaks in the interior. Saylor, Huang & Miller (1978) report that there are observations of oscillatory flows in the interior of Lake Michigan of frequency identical with that of the coastal internal Kelvin wave.

Internal Kelvin waves have been observed (Smith 1978) propagating poleward from the equator along the coasts of Ecuador and Peru. It has been postulated (see, for example, McCreary 1976; Philander 1979) that Kelvin waves play a prominent role in the adjustment process involved in the phenomenon known as 'El Niño' [Upwelling along the coast of Peru is seasonal with a considerable interannual variability. At infrequent intervals the upwelling is absent for an entire season, a natural catastrophe referred to as 'El Niño']. It is known that there is little change in local coastal forcing that could give rise to El Niño. It is therefore likely that the phenomenon is the result of some sudden change in the forcing conditions over the interior ocean far from the coast. This hypothesis is more plausible when the internal Kelvin waves are viewed as an ocean-wide mode with significant effects at the boundary, rather than purely an edge wave trapped by the boundary which would respond only to local conditions. We have shown that the velocity component of an internal Kelvin wave normal to the boundary is non-zero and hence could be instrumental in communicating the effects of an interior forcing to the coastal regions.

7. Discussion

The primary purpose of this paper is to study small-amplitude oscillatory motions in a uniformly rotating, density-stratified fluid inside a rigid container. We assume the fluid is Boussinesq and inviscid. We examine here only the situation when the rotation axis is parallel to gravity and the Brunt–Väisälä frequency N is constant. However, we investigate a variety of container shapes, and our results are not restricted to the long-wave or shallow-layer limits in which the vertical scale is much smaller than horizontal scales. We also consider arbitrary values for N and the angular rotation frequency f. We note that the response of the fluid to an arbitrary initial perturbation consists of both oscillatory motions and a time-independent quasi-geostrophic motion. The latter, steady motion is specified by the initial potential vorticity together with appropriate boundary conditions, leaving the oscillatory motions to be excited by the remainder of the initial disturbance.

Assuming the existence of square-integrable modes, we demonstrate a number of their properties and obtain explicit descriptions of the free modes associated with the cylinder and the sphere. In general there exist two distinct classes of oscillations which we have called class I and class II. The class I modes are inertia–gravity waves which are closely related to the inertial waves that exist in a homogeneous rotating fluid, and to internal waves that exist in a stratified non-rotating fluid. Inertial waves have been extensively examined and a summary of the results is found in Greenspan (1968). Inertial modes in a container satisfy Poincaré's problem,

$$\nabla^2\Phi - \frac{f^2}{\lambda^2}\Phi_{zz} = 0 \tag{7.1}$$

with

$$\hat{\mathbf{n}}.\nabla\Phi - \frac{f}{i\lambda}(\hat{\mathbf{n}}\times\hat{\mathbf{k}}).\nabla\Phi - \frac{f^2}{\lambda^2}(\hat{\mathbf{n}}.\hat{\mathbf{k}})\,\Phi_z = 0 \quad \text{on} \quad B. \tag{7.2}$$

Equations (7.1) and (7.2) are obtained from (2.21) and (2.22) by setting N^2 equal to zero. To emphasize the similarities and differences, we rewrite (2.21) and (2.22) as

$$\nabla^2\Phi - \frac{(N^2-f^2)}{\sigma^2}\,\Phi_{zz} = 0 \tag{7.3}$$

with

$$\hat{\mathbf{n}}.\nabla\Phi - \frac{f}{i(N^2-\sigma^2)^{\frac{1}{2}}}(\hat{\mathbf{n}}\times\hat{\mathbf{k}}).\nabla\Phi - \frac{N^2-f^2}{\sigma^2}(\hat{\mathbf{n}}.\hat{\mathbf{k}})\,\Phi_z = 0 \quad \text{on} \quad B, \tag{7.4}$$

where $\sigma^2 = N^2-\lambda^2$ (for convenience we have assumed $N^2 > f^2$; otherwise, $\sigma^2 = \lambda^2 - N^2$). In the case of axisymmetric modes where $(\hat{\mathbf{n}}\times\hat{\mathbf{k}}).\nabla\Phi$ is zero, (7.3) and (7.4) are exactly equivalent to (7.1) and (7.2) with f^2 replaced by $|N^2-f^2|$. Hence, the axisymmetric class I modes are exactly equivalent in structure to the axisymmetric inertial modes (recall property 5 of § 3 that there are no axisymmetric class II modes). The problem for non-axisymmetric class I modes is similar to the equivalent Poincaré problem, the only difference being the way in which the eigenvalue occurs in the coefficient of the second term in the boundary condition. This difference gives rise to a modification of the dispersion relation for the eigenvalues as functions of the wave-numbers. In a cylinder, the change is illustrated by comparing the coupled transcendental system (4.5) and (4.6) for λ_{mnk} and γ_{mnk} with the equivalent system with $N^2 = 0$. For a cylinder the frequency distribution of inertial modes for fixed wave-numbers m and k is shown in figure 3 by the intersection of the curves with the y axis. The introduction of stratification shifts the position of each value of λ^2, and the permissible range for λ^2 contracts. At the critical value N^2-f^2, the class I spectrum degenerates to a single point $\lambda^2 = f^2 = N^2$. This mode, with zero pressure gradient, has a structure, described in § 5, which is completely different from an inertial mode. As N^2 increases beyond f^2, the pattern is repeated nearly symmetrically, which is to be expected in view of the analogies between the case of pure rotation and pure stratification (see Veronis 1970). A novel feature is the appearance of a transition mode between the class I and class II modes at $\lambda^2 = f^2$.

In contrast with the class I modes, the class II low-frequency modes are not analogous to inertial modes. The existence of class II modes is crucially dependent on the presence of both rotation and stratification. These waves which have been referred to as internal Kelvin waves are asymmetric, propagate only in the direction of the mean rotation, and have a spatially decaying pressure field in the direction away from the boundary. The fact that (7.3) and (7.4) admit solutions of the class II type, while there are no analogous solutions to (7.1) and (7.2), can again be attributed to the particular way in which the eigenvalue enters the coefficient of the middle term in the boundary conditions (7.2) and (7.4). Thus, the subtle but important difference between the problem studied in this paper and Poincaré's problem gives rise to the distinct and novel features that are amongst our principal subjects of interest here. We remark that Malkus (1967) investigated hydromagnetic planetary waves, formulating the problem so that it became Poincaré's problem. In that hydromagnetic case, the problem is of interest because of the somewhat complicated relation between the wave frequency and the eigenvalue of Poincaré's problem. In contrast, our new results stem from the fact that our problem is not equivalent to Poincaré's problem.

As we noted in §2, results exist which suggest that the inertial-mode spectrum for a homogeneous fluid is continuous in certain geometries. For example, the energy of waves in a conical container appears to be funnelled into the apex, which results in a continuous spectrum of 'singular modes'. A problem of particular geophysical interest is that of inertial waves in a rotating spherical shell. Stewartson & Rickard (1969) use Laplace's tidal equations to study waves in a thin spherical shell. They conclude that the asymmetric modes are not square-integrable at so-called critical latitudes where the characteristics are tangent to the inner boundary. In the problems for a rotating stratified fluid we have described here, there is no evidence of singularities which would indicate that the modal spectrum is continuous. For class I modes, this is not surprising in view of the analogies with the modes of the homogeneous problem, for which the spectra for a cylinder and a sphere have been shown to be denumerable but dense (see Greenspan 1968). A continuous spectrum need not be expected for a smooth perturbation of such boundaries into other more singular boundaries. In fact, there is evidence from work of Stewartson & Walton (1976) that, even in a spherical shell, singularities at critical latitudes can be removed (or at least weakened) by the introduction of stratification. Further, London & Shen (1979) suggest that the critical-latitude singularity may be an artifact of the approximations implicit in the derivation of Laplace's tidal equations, in which the vertical structure of the governing equations is neglected. Since our work includes stratification and does not neglect vertical (radial) structure, we have no reason to be surprised at the absence of a continuous modal spectrum. For the case of class II modes, the possibility of a continuous spectrum may be ruled out in view of the general result for an elliptic partial differential equation with Dirichlet boundary conditions on a non-singular boundary (see Reed & Simon 1978).

In this present paper, we have made no restrictions on the ratio of N/f or on the vertical and horizontal length scales, except to require that the Froude number $F_R = f^2 L/g$ be small. Our results are therefore appropriate to a wide range of laboratory configurations. As noted in §2, there are geophysical contexts in which N/f could be as large as $O(10^2)$ or much less than unity, so from a physical point of view it is desirable to permit N/f to be unrestricted. A drawback of the present work lies in our assumption that the potential surfaces are always planes perpendicular to the axis of rotation. In applying our results to a lake or an ocean basin on the surface of a rotating sphere, the restrictions of our model demand that the sphere be replaced by a tangent plane and the horizontal component Ω_H of the earth's rotation be neglected. The neglect of Ω_H, sometimes called the traditional approximation (see Eckart 1960), can be justified under certain circumstances, for example when the vertical velocity is much less than the horizontal velocity. In his systematic derivation of Laplace's tidal equations, Miles (1974) justifies the neglect of Ω_H by making the assumption of a shallow ocean, i.e. $H/L \ll 1$. Hence, although we place no explicit restriction on H/L, we are implicitly assuming restrictions in invoking the traditional approximation in order to apply our results in a geophysical context. A further drawback of our model lies in the fact that we treat f as constant, whereas geophysically f varies with latitude ($f = 2\Omega \times \sin$ (latitude)). The neglect of the latitudinal variation of Coriolis parameter eliminates low-frequency Rossby waves. Obviously, a tangent plane approximation neglects effects of sphericity that may be important for geophysical or astrophysical models.

With geophysical and astrophysical applications in mind, we have extended some of the results of the present paper to waves in a rotating fluid stratified under more general gravitational fields. A radially symmetric field is a reasonable model for oceans, the atmosphere, or a planetary core, since it retains the sphericity of the potential surfaces fully. The traditional approximation, with its inherent restrictions, need not be invoked, and all latitudinal variations are retained. In Friedlander & Siegmann (1982), we show that the basic properties of oscillations described in §3 hold in this more general case. When the gravitational field is radial, the upper bound on λ^2 is increased to $f^2 + N^2_{\max}$. The modes can be classified as to type in a fashion analogous to our present classification. However, the situation becomes considerably more complicated, in that a wave of fixed frequency may resemble a class I wave (hyperbolic) in one spatial region, and a class II wave (elliptic) in another spatial region. This mixed behaviour leads to wave trapping which is absent from our present discussions. Further, when the restriction of small Froude number is relaxed, the potential surfaces are distorted by the centrifugal effect. Friedlander & Siegmann (1982) consider the case of vertical gravity plus centrifugal force. The major effects of the addition of the latter force are to increase the bound on λ^2, to decrease the width of the frequency range for internal Kelvin waves, and to admit the existence of mixed waves. Finally, we note that internal waves in more general gravitational fields are described by partial differential equations that are considerably more complicated than Poincaré's problem. The types of explicit solutions that have been examined in detail in this present paper, where curvature of the potential surfaces has been neglected, are evidently not available.

The authors wish to acknowledge stimulating conversations with L. N. Howard. The work of S. Friedlander was partially supported by N.S.F. Grants MCS 81-01698 and MCS 79-01718.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1965 *Handbook of Mathematical Functions*. Dover.

ALLEN, J. S. 1971 Some aspects of the initial value problem for the inviscid motion of a contained rotating, weakly stratified fluid. *J. Fluid Mech.* **46**, 1.

BRYAN, G. H. 1889 The waves on a rotating liquid spheroid of finite ellipticity. *Phil. Trans. R. Soc. Lond.* A **180**, 187.

CAIRNS, J. L. & WILLIAMS, G. O. 1976 Internal wave observations from a midwater float. 2. *J. Geophys. Res.* **81**, 1943.

CLARKE, A. J. 1977 Wind-forced linear and nonlinear Kelvin waves along an irregular coast line. *J. Fluid Mech.* **83**, 337.

CSANADY, G. T. 1976 Topographic waves in Lake Ontario. *J. Phys. Oceanog.* **6**, 93.

ECKART, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Pergamon.

ERDELYI, A. 1953 *Tables of integral transforms*. McGraw-Hill.

FRIEDLANDER, S. & SIEGMANN, W. L. 1982 Internal waves in a rotating stratified fluid in an arbitrary gravitational field. *Geophys. Astrophys. Fluid Dyn.* (to appear).

GARRETT, C. & MUNK, W. 1979 Internal waves in the ocean. *Ann. Rev. Fluid Mech.* **11**, 339.

GRADSHTEYN, I. S. & RYZHIK, I. W. 1965 *Table of Integrals, Series, and Products*. Academic.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

HAMBLIN, R. F. 1978 Internal Kelvin waves in a fjord lake. *J. Geophys. Res.* **83**, 2409.

HOWARD, L. N. & SIEGMANN, W. L. 1969 On the initial value problem for rotating stratified flow. *Stud. Appl. Math.* **48**, 153.

HUTHNANCE, J. M. 1978 On coastal trapped waves: analysis and numerical calculation by inverse iteration. *J. Phys. Oceanog.* **8**, 74.

KAMENKOVICH, V. M. 1977 *Fundamentals of Ocean Dynamics*. Elsevier.

KRAUSS, W. 1966 *Methoden und Ergebnisse der Theoretischen Oceanographie II: Interne Wellen.* Berlin: Gebrueder-Borntraeger.

KUDLICK, M. D. 1966 On transient motions in a contained rotating fluid. Ph.D. dissertation, Massachusetts Institute of Technology.

LEBLOND, P. H. & MYSAK, L. A. 1979 Ocean waves: a survey of some recent results. *S.I.A.M. Review* **21**, 289.

LONDON, S. & SHEN, M. C. 1979 Free oscillations in a rotating spherical shell. *Phys. Fluids* **22**, 2071.

MALKUS, W. V. R. 1967 Hydromagnetic planetary waves. *J. Fluid Mech.* **28**, 793.

MCCREARY, J. 1976 Eastern tropical ocean response to changing wind systems: with application to El Niño. *J. Phys. Oceanog.* **6**, 632.

MILES, J. W. 1974 On Laplace's tidal equations. *J. Fluid Mech.* **66**, 241.

MYSAK, L. A. 1980*a* Topographically trapped waves. *Ann. Rev. Fluid Mech.* **12**, 45.

MYSAK, L. A. 1980*b* Recent advances in shelf wave dynamics. *Rev. Geophys. Space Phys.* **18**, 211.

OU, H. W. & BENNETT, J. R. 1979 A theory of the mean flow driven by long internal waves in a rotating basin, with application to Lake Kinneret. *J. Phys. Oceanog.* **9**, 1112.

PHILANDER, S. G. H. 1978 Variability of the tropical oceans. *Dynamics Oceans & Atmos.* **3**, 191.

PHILLIPS, O. M. 1977 *The Dynamics of the Upper Ocean*. Cambridge University Press.

REED, M. & SIMON, B. 1978 *Methods of Modern Mathematical Physics*, vol. IV. Academic.

RELTON, F. E. 1946 *Applied Bessel Functions*. Blackie.

SAYLOR, J., HUANG, J. C. K. & MILLER, G. 1978 Observations of rotational waves in Southern Lake Michigan. *Trans. Am. Geophys. Union* **59** (12), 1107.

SMITH, R. L. 1978 Poleward propagating perturbations in currents and sea levels along the Peru coast. *J. Geophys. Res.* **83**, 6083.

SPIEGEL, E. A. & VERONIS, G. 1960 On the Boussinesq approximations for a compressible fluid. *Astrophys. J.* **131**, 442.

STEWARTSON, K. 1971 On trapped oscillations of a rotating fluid. *Tellus* **23**, 506.

STEWARTSON, K. & RICKARD, J. A. 1969 Pathological oscillations of a rotating stratified fluid. *J. Fluid Mech.* **35**, 759.

STEWARTSON, K. & WALTON, I. C. 1976 On waves in a thin shell of stratified rotating fluid. *Proc. R. Soc. Lond.* A **349**, 141.

VERONIS, G. 1970 The analogy between rotating and stratified fluids. *Ann. Rev. Fluid Mech.* **2**, 37.

WANG, D. P. & MOOERS, C. N. K. 1976 Coastal trapped waves in a continuously stratified ocean. *J. Phys. Oceanog.* **6**, 853.

WUNSCH, C. 1973 On the mean drift in large lakes. *J. Limnol. Oceanog.* **18**, 793.